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不同空间Julia集的非线性广义同步

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摘要: 基于利用线性耦合实现空间Julia 集线性广义同步的研究成果, 本文通过应用非线性反馈控制和引入变量变换的方法解决了空间Julia集的非线性广义同步问题. 首先基于变量变换, 得到误差空间Julia系统, 为了镇定该系统使其实现非线性广义同步, 给出了确定的非线性函数关系式. 另外, 根据空间Julia集的稳定区域, 解析地确定了实现非线性广义同步的耦合强度的稳定域. 然后, 给出了稳定的复不动平面与空间Julia集的同步的关系. 最后, 用一个例子验证了该解析方法是可行的.

关键词: 空间Julia集; 广义同步; 稳定区域; 复杂的

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Nonlinear generalized synchronization of two different spatial Julia sets

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Abstract: Referring to the realization of linear generalized synchronization for spatial Julia sets using linear coupling, we deal with its nonlinear generalized synchronization by applying nonlinear feedback control and introducing a multi-variable transform in this work. Based on this multi-variable transform, the error spatial Julia system is obtained and the nonlinear function is given to stabilize the error spatial Julia system such that the generalized synchronization will be achieved. Moreover, according to the stability domains given for spatial Julia sets, we determine analytically the stability domain of the coupling strength for the generalized synchronization. Then, the relationship between the complex stable fixed plane and the synchronization of the spatial Julia sets is also analyzed. Finally, an example is presented to validate the scheme and the analysis.

Key words: spatial Julia sets; generalized synchronization; stable domain; complex

1 Introduction

In 1975, the mathematician Benoit Mandelbrot introduced the word ‘Fractal’, meaning ‘broken’, to describe irregular and complicated natural phenomena such as coastlines, plant branching, and mountains that cannot be described via Euclidean geometry. Later, Fractal was a foremost phenomenon in nonlinear systems and formed the most classical contents of the nonlinear theories along with chaos and bifurcation theories. Recently, Fractal has been widely used in areas of the natural and social sciences[1–3]. For example, the fractal simulation has been used to produce a library of plant dynamic configuration in favor of botany via iterative function systems, L-systems, DLA models and so on[4–7]. According to the developments of genetics, molecular biology and biotechnology, the process of plant growth is simulated by computer, which may guide the plant breeding and shorten the breeding cycle.

Julia sets, one of the most important fractal sets, has received much more attention in its properties, its applications and its control. Beck[8] gave the physical meaning of Julia sets based on the deterministic version of the Langevin problem. The singularity spectra of the Julia sets of s-state potts model on the diamond hierarchical lattice was investigated[9]. Wang et al.[10] also discussed the physical meaning of Julia sets based on the particle dynamics characteristics and found that the change law of the particle’s velocity can be reflected visually by the fractal characteristics of generalized Julia sets. A lot of approaches have been proposed for realizing the control of Julia sets, which include feedback control, gradient control, optimal control and coupling control[11–13]. These controls satisfy extensive requirements of the engineering and achieve the desirable aims.

Obviously, the study of Julia sets has mainly focused on the one-dimensional Julia sets until now. In fact, the spatial Julia sets is a new and important research topic. The

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spatial Julia sets in 4–D quaternion was displayed firstly via boundary tracking method[14]. Since then, a deterministic 3–D Julia set has been constructed via the ray tracing and other 3–D Julia set was also produced based on customized complexified quaternion. Gintz et al.[15] introduced the complexified quaternion algebra to iteratively compute the boundaries of the spatial Julia sets so as to obtain the new structures of the fractal sets. Sui et al.[16] constructed the Julia sets in coupled map lattice and gave its some properties.

In fact, the spatial Julia sets have been developed well in the basic properties and the drawing of the graphics. Recently, the control has also been introduced to spatial Julia sets firstly via transforming matrix[17]. This control method is only a mathematic transform and has limits in engineering applications. Later, gradient control and optimal control were also used to achieve the control of the stable space of the fixed plane from Julia sets in coupled map lattice[18].

It is well known that synchronization has also been applied widely in the field of secure or private communications. More generally, it is an important phenomenon not only observed in nature, but also artificially induced, as it is of great concern for many problems of physics, applied sciences and engineering[18–19]. Without any exception, synchronization is also introduced to the spatial Julia sets. The synchronization of different Julia sets in coupled map lattice is reported firstly[18]. Later, we studied the linear generalized synchronization of spatial Julia sets from a complex iteration $z_{m+1,n}+az_{m,n+1}=(1+a)^2z^2_{m,n}+c$ (a is a real parameter and c is a complex constant)[20]. While, in many real cases and engineering applications, two different complex variables need to have nonlinear relationship.

Motivated by the above observation and inspired by our recent work[20], we discuss in this paper the nonlinear generalized synchronization of two different spatial Julia sets. Here we introduce the multi-variables transform such that the nonlinear relationship of two different complex variables is satisfied. So the error spatial Julia system is obtained and the nonlinear function is characterized to stabilize the error spatial Julia system such that the generalized synchronization is achieved.

The organization of this paper is as follows. In the next section, we introduce the definition of generalized synchronization of spatial Julia sets and accomplish the nonlinear generalized synchronization between two different spatial Julia sets. In Section 3, a numerical example is taken to verify the feasibility of nonlinear feedback controlled generalized synchronization. Section 4 analyzes the relationship between the stable fixed plane and nonlinear synchronization. Finally, conclusions are given in Section 5.

2 Generalized synchronization of the spatial Julia sets

In this section, the notion of generalized synchronization of the spatial Julia sets[20] is introduced as follows:

Definition 1 Consider two spatial Julia systems:

$$z_{m+1,n}+az_{m,n+1}=f(z_{m,n},w_{m,n};c_1,a), \tag{1}$$

$$w_{m+1,n}+aw_{m,n+1}=g(z_{m,n},w_{m,n};c_2,a), \tag{2}$$

where f and g are the nonlinear functions.

Adding a coupling term $p(z_{m,n},w_{m,n},c_2,a;L)$ to the system (1) yields

$$z_{m+1,n}+az_{m,n+1}=\\ f(z_{m,n},w_{m,n};c_1,a)+p(z_{m,n},w_{m,n},c_2,a;L), \tag{3}$$

where p is a coupling item about $z_{m,n},w_{m,n},c_2$ and a, and L is a coupling strength. Clearly, there exists a Julia set corresponding to every L. The Julia sets of (2) and (3) are denoted by J_1 and J_2, respectively.

If there exists a function $\psi(\cdot)$ such that the spatial Julia sets of $\psi(J_2)$ becomes the same with J_1 when L tends to L_0, namely,

$$\lim_{L=L_0}(J_2\cup\psi(J_1)\setminus J_2\cap\psi(J_1))=\varnothing, \tag{4}$$

for some L_0, the generalized synchronization of Julia sets of the systems (1) and (2) is achieved.

We take $f(z_{m,n},w_{m,n};c_1,a)=(1+a)^2z^2_{m,n}+c_1$ and $g(z_{m,n},w_{m,n};c_2,a)=(1+a)^2w^2_{m,n}+c_2$, then systems (1)−(3) reduce to

$$z_{m+1,n}+az_{m,n+1}=(1+a)^2z^2_{m,n}+c_1, \tag{5}$$

$$w_{m+1,n}+aw_{m,n+1}=(1+a)^2w^2_{m,n}+c_2, \tag{6}$$

and

$$z_{m+1,n}+az_{m,n+1}=\\ (1+a)^2z^2_{m,n}+c_1+p(z_{m,n},w_{m,n},c_2,a;L), \tag{7}$$

where c_1 and c_2 are complex parameters and $c_1\neq c_2$. It is obvious that the system (7) is driven by the system (6). Here we take the nonlinear feedback term or the coupling item as

$$p(z_{m,n},w_{m,n},c_2,a;L)=L[g(z_{m,n},w_{m,n})]^2, \tag{8}$$

and the $g(z_{m,n},w_{m,n})$ satisfies the following nonlinear relationship

$$g(z_{m,n},\,w_{m,n})=a_1z_{m,n}+a_2w^2_{m,n}+a_3, \tag{9}$$

which is different from the linear relationship in Ref.[20]. We introduce the error system

$$e_{m,n}=h(z_{m,n},w_{m,n}),$$

so two functions $h(z_{m,n},w_{m,n})$ and $g(z_{m,n},w_{m,n})$ are expected to ensure that error states $e_{m+1,n},\ e_{m,n+1}$ and $e_{m,n}$ satisfy the error spatial Julia system

$$e_{m+1,n}+ae_{m,n+1}=(1+a)^2e^2_{m,n}+c, \tag{10}$$

and $c\in(\boldsymbol{D}_1\cup\boldsymbol{D}_2)$($\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ described different stable domains respectively in Ref.[20]). Generally, there exists a functional relationship between c and the rest parameters c_1,c_2,L,a, namely, $c=\varphi(c_1,c_2,L,a)$.

According to Theorem 2 of Ref.[20], if we find an appropriate L such that the parameter c of system (10) satisfies $c\in(\boldsymbol{D}_1\cup\boldsymbol{D}_2)$, then the error state $e_{m,n}$ tends to the stable fixed plane $e^*=\dfrac{1\pm\sqrt{1-4c}}{2(1+a)}$, namely, $h(z_{m,n},w_{m,n})=$ constant, that is, $z_{m,n}$ and $w_{m,n}$ satisfy the nonlinear functional relationship. Hence, the nonlinear generalized synchronization of the spatial Julia system is achieved.

For simplicity only, the generalized synchronization of the spatial Julia system with $a \geqslant 0$ is discussed. The other two similar cases when $a < 0$ and $a \neq -1$ will be omitted.

Substituting (8) into (7), we get

$$\begin{cases} z_{m+1,n} + az_{m,n+1} = \\ (1+a)^2 z_{m,n}^2 + c_1 + L(a_1 z_{m,n} + a_2 w_{m,n}^2 + a_3)^2, \\ w_{m+1,n} + aw_{m,n+1} = (1+a)^2 w_{m,n}^2 + c_2, \end{cases} \tag{11}$$

where L is a coupling strength. Noted that the first subsystem of subsystem of the system (11) is driven by the second one.

To guarantee that systems (5) and (6) realize generalized synchronization, namely, the error state $e_{m,n}$ satisfies the error spatial Julia system (10), we consider the following multi-variables transform:

$$e_{p,q} = \alpha z_{p,q} + \beta a^{q-n} w_{p,q}^2 + (2\beta a w_{m,n+1} w_{m+1,n} + 1)^{p-m} + \gamma - 1, \tag{12}$$

where $p, q, m, n \in N_0 = \{0, 1, 2 \cdots\}$, with $p \geqslant m$ and $q \geqslant n$. Then

$$\begin{cases} e_{m+1,n} = \\ \alpha z_{m+1,n} + \beta w_{m+1,n}^2 + \\ 2\beta a w_{m,n+1} w_{m+1,n} + \gamma, \\ e_{m,n+1} = \alpha z_{m,n+1} + \beta a w_{m,n+1}^2 + \gamma, \\ e_{m,n} = \alpha z_{m,n} + \beta w_{m,n}^2 + \gamma. \end{cases} \tag{13}$$

It follows from (13) that

$$\begin{aligned} & e_{m+1,n} + ae_{m,n+1} = \\ & \alpha z_{m+1,n} + \beta w_{m+1,n}^2 + 2\beta a w_{m,n+1} w_{m+1,n} + \\ & \gamma + a(\alpha z_{m,n+1} + \beta a w_{m,n+1}^2 + \gamma) = \\ & \alpha(z_{m+1,n} + az_{m,n+1}) + \beta(w_{m+1,n} + \\ & aw_{m,n+1})^2. \end{aligned} \tag{14}$$

Substituting the third equation of (13) into the error spatial Julia system (10), we have

$$\begin{aligned} & e_{m+1,n} + ae_{m,n+1} = \\ & (1+a)^2(\alpha z_{m,n} + \beta w_{m,n}^2 + \gamma)^2 + c. \end{aligned} \tag{15}$$

From equations (14) and (15), we obtain

$$\begin{cases} \alpha = 1 + \dfrac{L}{(1+a)^2} a_1^2,\ \beta = \dfrac{L}{(1+a)^2} a_1 a_2, \\ \gamma = \dfrac{L}{(1+a)^2} a_1 a_3, \\ c = La_3^2 + \dfrac{La_1^2}{(1+a)^2} c_1 + c_1 + \\ \quad \dfrac{La_1 a_2}{(1+a)^2} c_2^2 + \dfrac{L}{1+a} a_1 a_3, \\ a_2 = -(1+a)^2 a_1,\ a_3 = -a_1 c_2. \end{cases} \tag{16}$$

Setting $a_1 = 1$ for simplicity, we get $a_2 = -(1+a)^2$, $a_3 = -c_2$ and

$$g(z_{m,n},\ w_{m,n}) = z_{m,n} - (1+a)^2 w_{m,n}^2 - c_2.$$

Then the coupled system (11) reduces to

$$\begin{cases} z_{m+1,n} + az_{m,n+1} = \\ (1+a)^2 z_{m,n}^2 + c_1 + L[z_{m,n} - \\ (1+a)^2 w_{m,n}^2 - c_2]^2, \\ w_{m+1,n} + aw_{m,n+1} = (1+a)^2 w_{m,n}^2 + c_2, \end{cases} \tag{17}$$

and α, β, γ and c become

$$\begin{cases} \alpha = 1 + \dfrac{L}{(1+a)^2}, \\ \beta = -L, \\ \gamma = -\dfrac{Lc_2}{(1+a)^2}, \\ c = c_1 + \dfrac{L}{(1+a)^2}[c_1 - (1+a)c_2]. \end{cases} \tag{18}$$

Hence, $c = c_1 + \dfrac{L}{(1+a)^2}[c_1 - (1+a)c_2]$ satisfies the condition of $c \in \boldsymbol{D}_1$(The stable domain $\boldsymbol{D}_1$ was shown in Fig.1(a) of Ref.[20]).

Based on the fourth formula of (18), we get

$$L = \frac{(1+a)^2}{c_1 - (1+a)c_2}(c - c_1). \tag{19}$$

Denote

$$\begin{aligned} & \frac{(1+a)^2}{c_1 - (1+a)c_2} = \\ & \frac{(1+a)^2}{|c_1 - (1+a)c_2|}(\cos\theta + \mathrm{i}\sin\theta), \end{aligned}$$

where

$$\theta = -\arctan\frac{\mathrm{Im}(c_1 - (1+a)c_2)}{\mathrm{Re}(c_1 - (1+a)c_2)}.$$

Then the formula (19) becomes

$$L = \frac{(1+a)^2}{|c_1 - (1+a)c_2|}(\cos\theta + \mathrm{i}\sin\theta)(c - c_1), \tag{20}$$

where $\dfrac{(1+a)^2}{|c_1 - (1+a)c_2|}$ is a telescopic multiple and $\cos\theta + \mathrm{i}\sin\theta$ is a rotation of θ degree. The stable domain $\boldsymbol{D}_1'$ of the coupling strength L is obtained after moving the domain $\boldsymbol{D}_1$ to the right c_1, becoming $\dfrac{(1+a)^2}{|c_1 - (1+a)c_2|}$ times of the domain $\boldsymbol{D}_1$ in size, and rotating through θ clockwise. Taking $a = 0.5$, $c_1 = 0.5\mathrm{i}$ and $c_2 = -0.5 + 0.5\mathrm{i}$, we get the domain $\boldsymbol{D}_1'$ of the coupling strength L as shown in Fig.1.

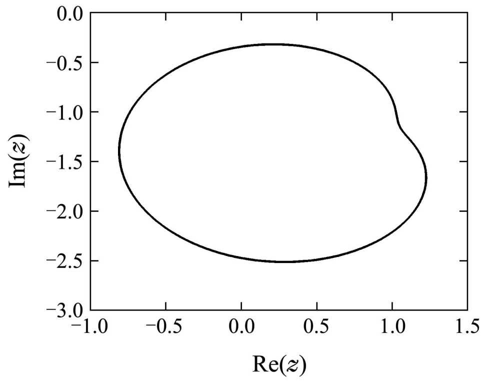


Fig. 1 The stability domain $\boldsymbol{D}_1'$ with $a \geqslant 0$

The nonlinear generalized synchronization of the systems (5) and (6) is discussed as follows. Substituting α, β and γ of (18) into the third equation of (13), and letting $M = \dfrac{L}{(1+a)^2}$, we obtain

$$e_{m,n}=(1+M)z_{m,n}-M(1+a)^2w_{m,n}^2-Mc_2,$$

namely,

$$w_{m,n}^2=\frac{1+M}{M(1+a)^2}z_{m,n}-\frac{1}{M(1+a)^2}e_{m,n}-\frac{c_2}{(1+a)^2}.$$

Substituting $w_{m,n}^2$ into the coupled subsystem of (17), we have

$$\begin{aligned}&z_{m+1,n}+az_{m,n+1}=\\&\frac{1+M}{M}(1+a)^2(z_{m,n}-\frac{e_{m,n}}{1+M})^2+\\&\frac{(1+a)^2}{1+M}e_{m,n}^2+c_1.\end{aligned}\tag{21}$$

From systems (10)(21) and the forth formula of equation (18), we get

$$\begin{aligned}&z_{m+1,n}-\frac{1}{1+M}e_{m+1,n}+\\&a(z_{m+1,n}-\frac{1}{1+M}e_{m+1,n})=\\&\frac{1+M}{M}(1+a)^2(z_{m,n}-\frac{1}{1+M}e_{m,n})+\\&\frac{M}{1+M}(1+a)c_2.\end{aligned}\tag{22}$$

Similarly, we let

$$W_{m,n}=z_{m,n}-\frac{1}{M+1}e_{m,n},\tag{23}$$

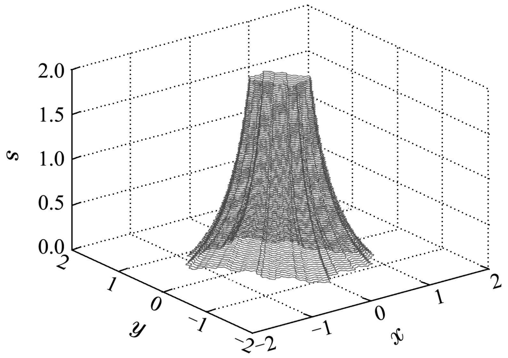
namely,

$$\begin{cases}W_{m+1,n}=z_{m+1,n}-\dfrac{1}{M+1}e_{m+1,n},\\ W_{m,n+1}=z_{m,n+1}-\dfrac{1}{M+1}e_{m,n+1}.\end{cases}$$

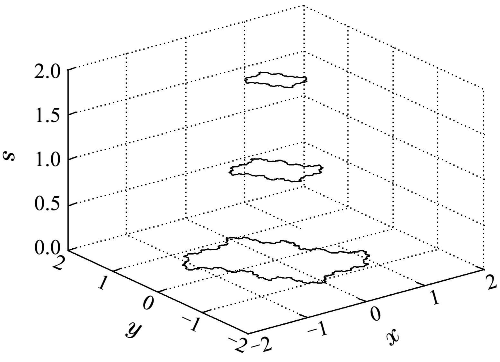
Then the coupled system (22) becomes

$$\begin{aligned}&W_{m+1,n}+aW_{m,n+1}=\\&\frac{1+M}{M}(1+a)^2W_{m,n}^2+\frac{(1+a)c_2}{(M+1)/M}.\end{aligned}\tag{24}$$

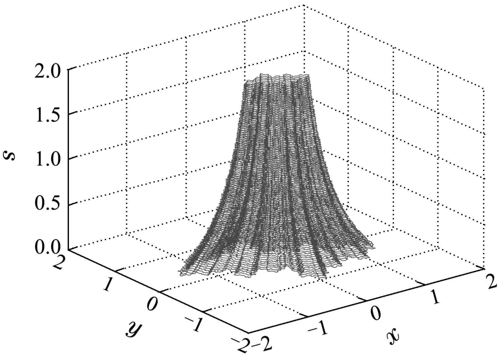
The system (24) is the same with the system (33) of Ref.[20]. The spatial Julia sets from the the coupled system (24) have been analysised detailedly in Ref.[20]. Here we will not repeat discussion about the spatial Julia set obtained by the nonlinear generalized synchronization.



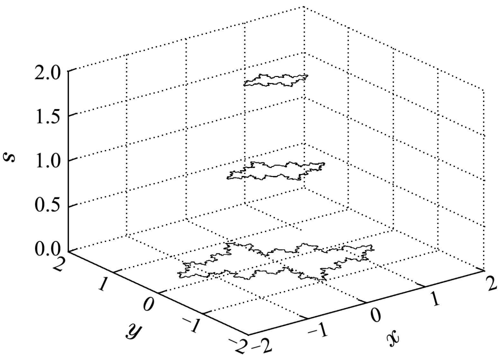
(a) The spatial Julia set of system (5) with $c_1=0.5\mathrm{i}$, $a\geqslant 0$



(b) The cross-section with $c_1=0.5\mathrm{i}$ and $a=0,\ 0.5,\ 1.5$



(c) The spatial Julia set of system (6) with $c_2=-0.5+0.5\mathrm{i}$, $a\geqslant 0$



(d) The cross-section with $c_2=-0.5+0.5\mathrm{i}$ and $a=0,\ 0.5,\ 1.5$

Fig. 2 The spatial Julia sets of systems (5) and (6) and their cross-sections

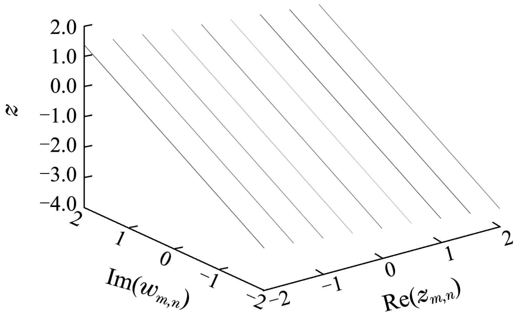
3 Applications

The generation of spatial Julia sets has been given in Ref.[20]. Similarly, we fixed $t=t_0=0$ in this paper, then we consider the points $(x,y,s,0)$ of the spatial Julia sets. Thereby, the spatial Julia sets are depicted in three-dimensional space R^3. We take $c_1=0.5\mathrm{i}$ and $c_2=-0.5+0.5\mathrm{i}$ in the systems (5) and (6).

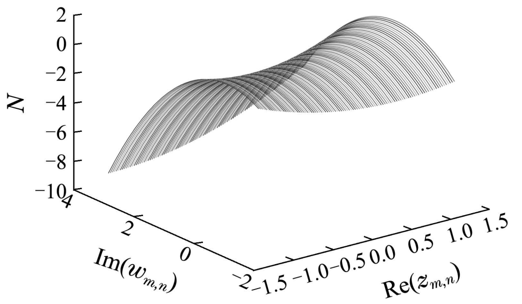
Definition 2.2 and Lemma 2 of Ref.[20] imply that the two systems satisfy the condition of the spatial Julia sets as shown in Figs.2(a) and 2(c)($a\geqslant 0$). Their cross-sections are also depicted in Figs.2(b) and 2(d) with $a=0,0.5,1.5$. Since the parameter $c\in\boldsymbol{D}_1, a\geqslant 0$ and the forth formula of (18), we take $a=0.5$ and obtain an inequality

$$|c_1+\frac{L}{(1+0.5)^2}[c_1-(1+0.5)c_2]|<0.67.$$

Hence, the states $z_{m,n}$ and $w_{m,n}$ satisfy a determinate nonlinear relationship as shown in Fig.3.



(a) The linear relationship figure of $w_{m,n}$ by $\mathrm{Re}(z_{m,n})$ and $\mathrm{Im}(w_{m,n})$



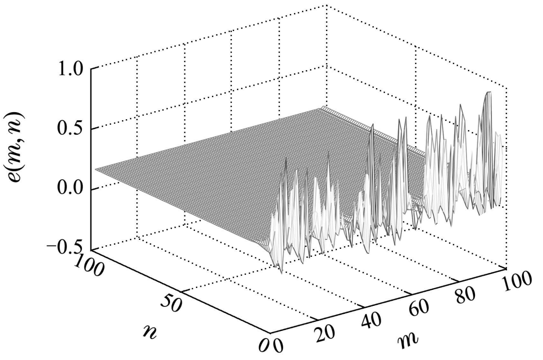
(b) The nonlinear relationship figure of $z_{m,n}$ by $\mathrm{Re}(z_{m,n})$ and $\mathrm{Im}(w_{m,n})$

Fig. 3 The nonlinear relationship of $z_{m,n}$ and $w_{m,n}$

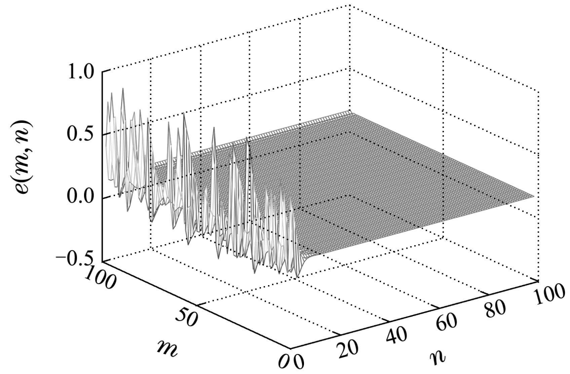
Here $c = 0.2(a = 0.5,\ L = 0.99 - 1.17\mathrm{i})$, the system (10) becomes

$$e_{m+1,n} + 0.5e_{m,n+1} = 2.25e_{m,n}^2 + 0.2. \qquad (25)$$

Therefore, $z_{m,n}$ and $w_{m,n}$ achieve generalized synchronization for sufficiently large m and n as shown in Figs.4 and 5. Fig.4(a) shows the dynamic behavior of $e_{m,n}$ with the change of m. Similarly, Fig.4(b) shows it with the change of n. When $a = 0, 1, 1.5$, the cross-section of Fig.5(a) is Fig.5(b). These cross-sections given in Figs.2(b)(d) and Fig.5(b) show that the Julia sets decreases gradually with the increase of its parameter a. From Fig.5, the position of the spatia Julia sets obtained is just transformed. The coefficient $\dfrac{M+1}{M}$ make the spatial Julia set of the system (24) dwindle and rotate, where $M = \dfrac{L}{(1+a)^2} = 0.44 - 0.52\mathrm{i}$. M is determined by L, when a is certain.

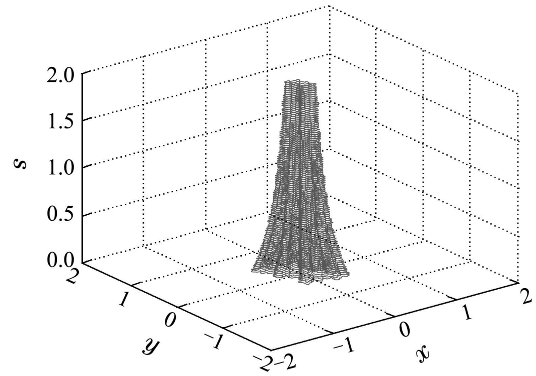


(a) Dynamic behavior of $e_{m,\,n}$ with the change of m

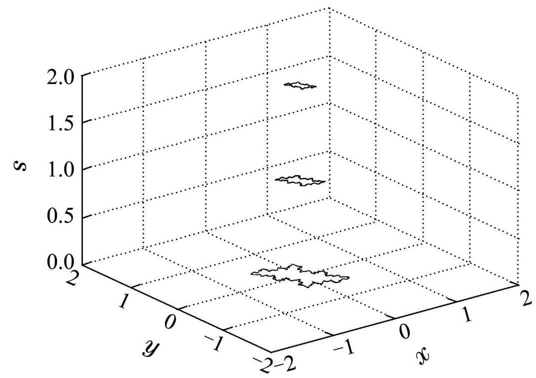


(b) Dynamic behavior of $e_{m,\,n}$ with the change of n

Fig. 4 $z_{m,n}$ and $w_{m,\,n}$ achieve generalized synchronization



(a) The spatial Julia set of generalized synchronization



(b) The cross-section of the spatial Julia set with $a = 0,\ 0.5,\ 1.5$

Fig. 5 The spatial Julia set from the nonlinear generalized synchronization between systems (5) and (6)

The discussion of the spatial Julia sets in the other cases as $x = x_0$ or $y = y_0$ or $s = s_0$ is similar and will thus be omitted.

4 The relationship between the complex fixed plane and synchronization

According to Theorem 1 of Ref.[20], we obtain the fixed plane $e_1^* = 0.48$ and $e_2^* = 0.18$ of system (25). Theorem 2 from Ref.[20] implies that the parameter $c \in \boldsymbol{D}_1$, then there exists a stable fixed plane of system (25). In the light of Lemma 1 of Ref.[20], we know that e_2^* is a stable fixed plane, which is confirmed by Fig.6. From Fig.7, system (25) is controlled to its stable fixed plane $e_2^* = 0.18$ with the increasing of n.

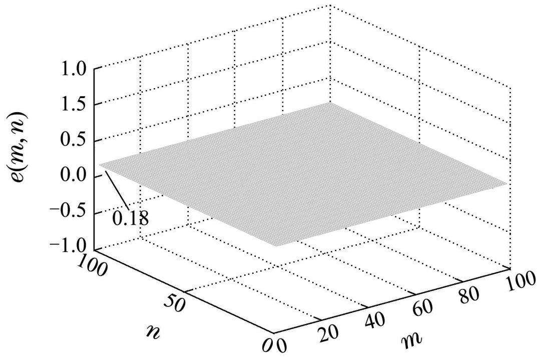


Fig. 6 The stable fixed plane of system (25)

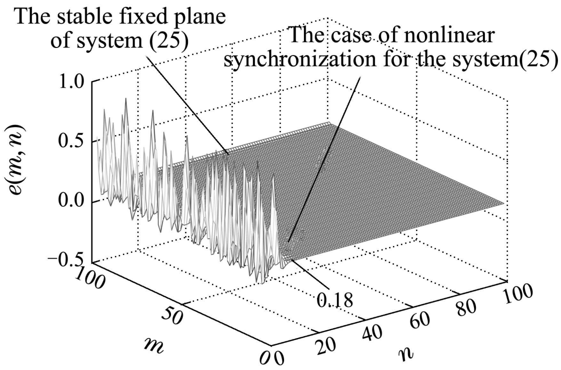


Fig. 7 System (25) is controlled to the fixed plane

5 Conclusions

In this paper, based on the basic theory in [20], we have achieved nonlinear generalized synchronization of two different spatial Julia sets via nonlinear feedback control method and introducing a multi-variables transform. At the same time, we have also specified the stable domains of the coupling strength. The numerical example has shown that the proposed scheme and analysis are effective and feasible.

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