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时滞耦合和非时滞耦合的奇异复杂动态网络之同步性准则

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摘要: 本文利用李雅普诺夫稳定性理论, 对时滞耦合和非时滞耦合的奇异复杂动态网络之同步获得了一些新的充分条件. 这些条件均可转化为求解一组线性矩阵不等式(LMI). 在降低准则保守性的过程中, 本文充分运用了矩阵函数的凸性和自由权重矩阵理论. 最后给出了两个数例; 与已有文献做了比较, 说明本文结论的有效性, 以及较低的保守性.

关键词: 同步性; 复杂动态网络; 奇异系统; 时滞依赖标准; 线性矩阵不等式

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Synchronization criteria for singular complex dynamical networks with delayed coupling and non-delayed coupling

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Abstract: By using Lyapunov-Krasovskii (LK) functional approach, we derive novel synchronization criteria in the form of linear matrix inequalities (LMIs) for singular complex dynamical networks with delayed coupling and non-delayed coupling. The convexity of matrix functions and the free-weighting matrix method are fully exploited to reduce the conservatism of the results we obtained. Numerical examples are presented to illustrate the efficiency and less conservatism of the proposed method.

Key words: synchronization; complex dynamical networks; singular system; delay-dependent criteria; linear matrix inequality

1 Introduction

In recent years, complex networks have received increasing attentions. Many practical systems can be modeled by complex networks[1–14]. There are two kinds of behaviors in complex network: static and dynamical behaviors. Obviously, many of these networks show out complexity in the overall topological and dynamical properties of the network nodes and the coupled units. Among these properties, people especially pay their attentions to the synchronization problem of complex dynamical networks[3–7]. The synchronization of general networks with state time-delays and coupling time-delays has been considered extensively[8–14]. Very recently, in order to obtain less conservative conditions, some new methods and techniques have been used, such as the free matrix method, delay decomposition, a piecewise analysis method, and so on, see [15–21]. To the best of the authors' knowledge, the method of dividing delay is the best one to handle the stability of system with delay, by which the result near analytical delay limit can be obtained in [18–21]. However, it should be noticed that most of the studies on synchronization of dynamical network in the above articles were actually performed under some implicit assumptions that there exists the information communication of nodes by the edges either at t or time $t-h$. The authors of [22–25] pointed that in many circumstance, this simplification does not match satisfactorily the peculiarities of real networks. There exists the information communication of nodes not only at t but also at time $t-h$, whereas the synchronization of both delay-coupled and non-delay-coupled complex dynamical network almost been ignored in the literatures[25]. Therefore, synchronization of complex networks with non-delayed and delayed coupling are extensively investigated in [22–25].

In the past decades, the studies on singular systems have been made great progress. It is well known that the singular systems can describe physical systems better than the regular (nonsingular) ones and they are extensive applied in control engineering: such as circuits, mechanical systems, economics, etc.[26–35]. S. Y. Xu, et al.[28] pointed that singular systems can be introduced to improve the traditional complex networks describe the singular dynamic behaviors of nodes. Many results of regular systems have been extended to singular cases, e. g., [29–30], where the robust stability and generalized quadratic stability were investigated via LMI approach. Also, the singu-

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lar systems with time delay[31–32] were discussed very extensively. Moreover, the singular hybrid coupled network systems are introduced to describe complex dynamical networks in [33], which give a more general description of physical systems than the normal one. The synchronization of singular complex dynamical networks with coupling delays is considered in [33–35]. In [34] a sufficient condition for global synchronization was derived by developing a strict linear matrix inequality (LMI) designed approach for singular complex dynamical network with coupling constant time delays; and [35] proposed a synchronization criterion based on LMI that are easily-solvable for that with coupling time-varying delays. However, in view of the analysis in the first paragraph, it is necessary to consider the synchronization of both delay-coupled and non-delay-coupled singular complex dynamical network. As far as we know, few literatures involves in this topic yet.

This paper proposes a synchronization criterion for delay-coupled and non-delay-coupled singular complex dynamical network based on LMIs, which are easily-solvable. In order to reduce the conservativeness of the criteria, modified Lyapunov-Krasovskii functions and some known-techniques, such as integral inequality and a piecewise analysis method, etc., are applied in this paper. Some illustrative examples are provided to show the effectiveness and advantage of the new criteria by comparing with the recently reported results.

Notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\boldsymbol{P}\in\mathbb{R}^{n\times n}$, $\boldsymbol{P}>0$(or $\boldsymbol{P}<0$) mean that $\boldsymbol{P}$ is a positive (or negative) definite matrix, respectively. $\boldsymbol{I}$ and 0 are an identity matrix and a null matrix with appropriate dimension, and $\text{diag}\{a_1,a_2,\cdots,a_n\}$ denotes a n-order diagonal matrix. For a real matrix $\boldsymbol{B}$ and two real symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ of appropriate dimensions, $\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ * & \boldsymbol{C}\end{pmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry, and $\|\cdot\|$ denotes 2-norm throughout the paper.

2 Singular complex dynamical networks model and preliminaries

Consider time-varying delayed singular complex dynamical networks consisting of N identical nodes, in which each node is an n-dimensional dynamical subsystem:

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{x}}_i(t)=\ &\boldsymbol{A}\boldsymbol{x}_i(t)+\boldsymbol{f}(\boldsymbol{x}_i(t),t)+c_1\sum_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)+\\ &c_2\sum_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-h(t)),\ t>0,\ i=1,\cdots,N,\end{aligned}\tag{1}$$

where $\boldsymbol{E}\in\mathbb{R}^{n\times n}$ is a singular matrix, and $\text{rank}(\boldsymbol{E})=r(0<r<n)$. $\boldsymbol{x}_j(t)\in\mathbb{R}^n$ is the i-th state vector, $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is a constant matrix, $c_k>0(k=1,2)$ are positive constant which are coupling strength, $\boldsymbol{\Gamma}_k=\text{diag}\{\tau_{k1},\cdots,\tau_{kn}\}(k=1,2)$ are constant diagonal inner-coupling matrices. $\boldsymbol{G}=(g_{ij})_{N\times N}(i=1,2,\cdots,N)$ is the outer-coupling matrix representing the topological structure of the complex networks, in which g_{ij} is defined as follows: if there is a connection between node i and node $j(i\neq j)$, then $g_{ij}=g_{ji}=1$; otherwise, $g_{ij}=g_{ji}=0(i\neq j)$. The row sums of $\boldsymbol{G}$ are zero, i.e., $\sum\limits_{j=1,j\neq i}^{N}g_{ij}=-g_{ii},i=1,\cdots,N$. The nonlinear function $\boldsymbol{f}(\boldsymbol{x}_i(t),t)$ is globally Lipschitz, i. e.,

$$\begin{aligned}&\|\boldsymbol{f}(\boldsymbol{x}_i(t),t)-\boldsymbol{f}(\boldsymbol{s}(t),t)\|\leqslant\\ &l_i\|\boldsymbol{x}_i(t)-\boldsymbol{s}(t)\|,\ i=1,2,\cdots,N,\end{aligned}\tag{2}$$

where l_i is a nonnegative constant.

Let $C([-H,0],\mathbb{R}^n)$ be the Banach space of continuous functions that map the interval $[0,h]$ to $\mathbb{R}^n$, with norm $\|\varphi\|=\sup\limits_{-h\leqslant\theta\leqslant0}\|\varphi(\theta)\|$. The initial conditions of the functional differential Eq.(1) are given by $\boldsymbol{x}_i(t)=\varphi_i(t)\in C([-H,0],\mathbb{R}^n)$. It is assumed that Eq.(1) has a unique solution for these initial conditions[24].

Remark 1 Network (1) is a singular complex network model with both non-delayed coupling and delayed coupling. It means from [22–25] that the information of each node communicates with other nodes is at time t as well as at time $t-h$. In fact, this phenomenon exists in real world, for example, in the stock market, decision-making of trade-offs is impacted by the information at time t as well as at time $t-h$. It is obviously, the error considered the information communication with both nodes at time t and at time $t-h$ is much smaller than that considered only one of them in [33–35].

The following definition and lemmas are indispensable in deriving the proposed stability criterion, and they are stated below:

Lemma 1[3] The eigenvalues of an irreducible matrix $\boldsymbol{G}=(g_{ij})\in\mathbb{R}^{N\times N}$ with $\sum\limits_{j=1,j\neq i}^{N}g_{ij}=-g_{ii},i=1,2,\cdots,N$ satisfy the following properties:

i) Real parts of all eigenvalues of $\boldsymbol{G}$ are less than or equal to 0 with multiplicity 1;

ii) $\boldsymbol{G}$ has an eigenvalue 0 with multiplicity 1 and the right eigenvector $(1,1,\cdots,1)^{\mathrm{T}}$.

Definition 1[33] Dynamical network (1) is said to achieve global (asymptotically) synchronization if

$$\lim_{t\to\infty}\|\boldsymbol{x}_i(t)-\boldsymbol{s}(t)\|=0,\ i=1,2,\cdots,N,\tag{3}$$

where $\boldsymbol{s}(t)\in\mathbb{R}^n$ may be an equilibrium point or a periodic orbit with $\boldsymbol{s}(t)=\dfrac{1}{N}\sum\limits_{i=1}^{N}\boldsymbol{x}_i(t)$. Let the error be $\boldsymbol{e}_i(t)=\boldsymbol{x}_i(t)-\boldsymbol{s}(t)$. One arrives at the error dynamical networks

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{e}}_i(t)=\ &\boldsymbol{A}\boldsymbol{e}_i(t)+\boldsymbol{F}_i(\boldsymbol{e}_i(t),t)+c_1\sum_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)+\\ &c_2\sum_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-h(t)),\end{aligned}\tag{4}$$

where

$$\boldsymbol{F}_i(\boldsymbol{e}(t),t)=\boldsymbol{f}(\boldsymbol{x}_i(t),t)-\boldsymbol{f}(\boldsymbol{s}(t),t),$$

$$\boldsymbol{f}(\boldsymbol{s}(t),t)=\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{f}(\boldsymbol{x}_i(t),t).$$

Model (4) can be written as compact form:

$$\boldsymbol{E}\dot{\boldsymbol{e}}(t)=\boldsymbol{A}\boldsymbol{e}(t)+\boldsymbol{F}(\boldsymbol{e}(t),t)+c_1\boldsymbol{\Gamma}_1\boldsymbol{e}(t)\boldsymbol{G}^{\mathrm{T}}+$$

$$c_2\boldsymbol{\Gamma}_2\boldsymbol{e}(t-h(t))\boldsymbol{G}^{\mathrm{T}}, \tag{5}$$

where

$\boldsymbol{e}(t)=(\boldsymbol{e}_1(t),\boldsymbol{e}_2(t),\cdots,\boldsymbol{e}_N(t))$,

$\boldsymbol{F}(\boldsymbol{e}(t),t)=(\boldsymbol{F}_1(\boldsymbol{e}(t),t),\boldsymbol{F}_2(\boldsymbol{e}(t),t),\cdots,\boldsymbol{F}_N(\boldsymbol{e}(t),t))$.

By the properties of the out-coupling matrix $\boldsymbol{G}$, there exists an unitary matrix $\boldsymbol{U}=[\boldsymbol{U}_1\ \ \boldsymbol{U}_2\ \cdots\ \boldsymbol{U}_N]\in\mathbb{R}^{N\times N}$ such that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{G}=\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}}$, with $\boldsymbol{\Lambda}=\mathrm{diag}\{\lambda_1,\lambda_2,\cdots,\lambda_N\}$ and $\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}}=\boldsymbol{I}$. Using the nonsingular transform $\boldsymbol{e}(t)\boldsymbol{U}=\boldsymbol{z}(t)=[\boldsymbol{z}_1(t)\ \ \boldsymbol{z}_2(t)\ \cdots\ \boldsymbol{z}_N(t)]\in\mathbb{R}^{n\times N}$, from Eq.(5), it follows the matrix equation:

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{z}}(t)=\ &\boldsymbol{A}\boldsymbol{z}(t)+\boldsymbol{F}(\boldsymbol{e}(t),t)\boldsymbol{U}+c_1\boldsymbol{\Gamma}_1\boldsymbol{z}(t)\Lambda+\\ &c_2\boldsymbol{\Gamma}_2\boldsymbol{z}(t-h(t))\Lambda.\end{aligned} \tag{6}$$

Equivalently, model (6) can be written as

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{z}}_i(t)=\ &(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)\boldsymbol{z}_i(t)+\boldsymbol{g}_i(t)+\\ &c_2\lambda_i\boldsymbol{\Gamma}_2\boldsymbol{z}_i(t-h(t)),\ i=1,2,\cdots,N.\end{aligned} \tag{7}$$

Here $\boldsymbol{g}_i(t)=\boldsymbol{F}(\boldsymbol{e}(t),t)\boldsymbol{U}_i$.

Thus, we have transformed the synchronization problem of the singular complex dynamical networks 1 into the synchronization problem of the N pieces of the corresponding error dynamical network (7). Note that $\lambda_1=0$ and $\boldsymbol{z}_1(t)=\boldsymbol{e}(t)\boldsymbol{U}_1=0$ from Lemma 1. Therefore, if the following $N-1$ pieces of the corresponding error dynamical network

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{z}}_i(t)=\ &(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)\boldsymbol{z}_i(t)+\boldsymbol{g}_i(t)+\\ &c_2\lambda_i\boldsymbol{\Gamma}_2\boldsymbol{z}_i(t-h(t)),\ i=2,\cdots,N\end{aligned} \tag{8}$$

are asymptotically stable, which implies that the synchronized states 1 are asymptotically stable.

Remark 2 In this paper, all synchronization criteria are derived based on the corresponding error dynamical network (8). In this mean, the outer coupling matrix $\boldsymbol{G}$ is assumed to satisfy Lemma 1, which may be some weak conditions, such as symmetric and diagonalizable. The case of $\boldsymbol{G}$ being not suitable for Lemma 1 may be an interested topic in our future work.

Definition 2[36] 1) The pair $(\boldsymbol{E},\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)$ is said to be regular if $\det(a\boldsymbol{E}-(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1))$ is not identically zero.

2) The pair $(\boldsymbol{E},\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)$ is said to be impulse free if $\deg(\det(a\boldsymbol{E}-(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)))=\mathrm{rank}\,\boldsymbol{E}$.

Lemma 2[37] The pair $(\boldsymbol{E},\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)$ is regular and impulse free if and only if there exist matries $\boldsymbol{P}_i$ such that the following inequalities hold for $i=2,\cdots,N$: $\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_i=\boldsymbol{P}_i\boldsymbol{E}\geqslant 0$ and $(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i^{\mathrm{T}}(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)<0$.

Lemma 3[38] If for any constant matrix $\boldsymbol{R}\in\mathbb{R}^{m\times m}$, $\boldsymbol{R}=\boldsymbol{R}^{\mathrm{T}}>0$, scalar $\gamma>0$ and a vector function $\varphi:[0,\gamma]\longrightarrow\mathbb{R}^m$ such that the integrations concerned are well defined, the following inequality holds:

$$-\gamma\int_{t-\gamma}^{t}\dot{\varphi}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\varphi}(s)\mathrm{d}s\leqslant$$

$$\begin{pmatrix}\varphi(t)\\ \varphi(t-\gamma)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}-\boldsymbol{R} & \boldsymbol{R}\\ * & -\boldsymbol{R}\end{pmatrix}\cdot\begin{pmatrix}\varphi(t)\\ \varphi(t-\gamma)\end{pmatrix}.$$

Lemma 4 Suppose that $h_1\leqslant h(t)\leqslant h_2$, where $h(t):\mathbb{R}_+\longrightarrow\mathbb{R}_+$. Then, for any $\boldsymbol{R}=\boldsymbol{R}^{\mathrm{T}}>0$, singular matrix $\boldsymbol{E}$, and free matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the following integral inequality holds:

$$\begin{aligned}&-\int_{t-h_2}^{t-h_1}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s\leqslant\\ &\boldsymbol{\zeta}^{\mathrm{T}}(t)((h(t)-h_1)\times\boldsymbol{X}\boldsymbol{R}^{-1}\boldsymbol{X}^{\mathrm{T}}+\\ &(h_2-h(t))\boldsymbol{Y}\boldsymbol{R}^{-1}\boldsymbol{Y}^{\mathrm{T}}+[\boldsymbol{X}\ \ \boldsymbol{Y}-\boldsymbol{X}\ \ -\boldsymbol{Y}]\boldsymbol{E}+\\ &\boldsymbol{E}^{\mathrm{T}}[\boldsymbol{X}\ \ \boldsymbol{Y}-\boldsymbol{X}\ \ -\boldsymbol{Y}]^{\mathrm{T}})\boldsymbol{\zeta}(t),\end{aligned} \tag{9}$$

where $\boldsymbol{\zeta}(t)=(\boldsymbol{x}^{\mathrm{T}}(t-h_1)\boldsymbol{x}^{\mathrm{T}}(t-h(t))\boldsymbol{x}^{\mathrm{T}}(t-h_2))^{\mathrm{T}}$, $\boldsymbol{X}=(\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{X}_2^{\mathrm{T}}\boldsymbol{X}_3^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{Y}=(\boldsymbol{Y}_1^{\mathrm{T}}\boldsymbol{Y}_2^{\mathrm{T}}\boldsymbol{Y}_3^{\mathrm{T}})^{\mathrm{T}}$.

Proof See Appendix I.

Lemma 5[39] Suppose that $a\leqslant h(t)\leqslant b$ and $\boldsymbol{Q}_i(i=1,2,3)$ are some constant matrices with appropriate dimensions. Then, $\boldsymbol{Q}_1+(h(t)-a)\boldsymbol{Q}_2+(b-h(t))\boldsymbol{Q}_3<0$ holds if and only if the following inequalities hold $\boldsymbol{Q}_1+(b-a)\boldsymbol{Q}_2<0$ and $\boldsymbol{Q}_1+(b-a)\boldsymbol{Q}_3<0$.

Lemma 6[40] Let $\boldsymbol{F}_0,\cdots,\boldsymbol{F}_p$ be quadratic function of the variable $\boldsymbol{x}\in\mathbb{R}^n$: $\boldsymbol{F}_i(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{T}_i\boldsymbol{x}+2\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{v}_i$, $i=0,\cdots,p$, where $\boldsymbol{T}_i=\boldsymbol{T}_i^{\mathrm{T}}$. We consider the following condition on

$$\begin{aligned}&\boldsymbol{F}_0,\cdots,\boldsymbol{F}_p:\boldsymbol{F}_0(\boldsymbol{x})\geqslant 0,\ \forall\boldsymbol{x}\\ &\text{s.t. }\boldsymbol{F}_i(\boldsymbol{x})\geqslant 0,\ i=0,\cdots,p.\end{aligned} \tag{10}$$

Obviously if there exist $\alpha_i\geqslant 0(i=0,\cdots,p)$ s.t.

$$\boldsymbol{F}_0(\boldsymbol{x})-\sum_{i=1}^{p}\alpha_i\boldsymbol{F}_i(\boldsymbol{x})\geqslant 0,\ \forall\boldsymbol{x},$$

then Eq.(10) holds.

3 Synchronization criteria for singular complex dynamical networks

In this section, we will investigate the stability problem of the error dynamical network system (8). Consequently, several criteria will be derived to show the impacts of the time-varying delay on the stability of the system. The derived criteria are delay-dependent. Now, we define that

$$\begin{aligned}&\boldsymbol{\xi}_i^{\mathrm{T}}(t)=\\ &[\boldsymbol{z}_i^{\mathrm{T}}(t)\ \ \boldsymbol{z}_i^{\mathrm{T}}(t-h(t))\ \ \boldsymbol{z}_i^{\mathrm{T}}(t-\frac{h}{2})\ \ \boldsymbol{z}_i^{\mathrm{T}}(t-h)\ \ \boldsymbol{g}_i^{\mathrm{T}}(t)],\end{aligned} \tag{11}$$

$$\boldsymbol{\eta}_i=[\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1\ \ c_2\lambda_i\boldsymbol{\Gamma}_2\ \ 0\ \ 0\ \ \boldsymbol{I}], \tag{12}$$

$$\boldsymbol{E}\dot{\boldsymbol{z}}_i(t)=\boldsymbol{\eta}_i\boldsymbol{\xi}_i(t). \tag{13}$$

From inequality (2), the Lipchitz condition for the nonlinear $\boldsymbol{g}_i(t)$ satisfies that[35]

$$\begin{aligned}\|\boldsymbol{g}_i(t)\|=\ &\|\sum_{k=1}^{N}[\boldsymbol{f}(\boldsymbol{x}_k(t),t)-\boldsymbol{f}(\boldsymbol{s}(t),t)]u_{ik}\|\leqslant\\ &\sum_{k=1}^{N}\|[\boldsymbol{f}(\boldsymbol{x}_k(t),t)-\boldsymbol{f}(\boldsymbol{s}(t),t)]\||u_{ik}|\leqslant\\ &\sum_{k=1}^{N}l\|\boldsymbol{x}_k(t)-\boldsymbol{s}(t)\|=\sum_{k=1}^{N}l\|\boldsymbol{e}_k(t)\|=\\ &\sum_{k=1}^{N}l\|\boldsymbol{z}(t)u_k^{\mathrm{T}}\|\leqslant\sum_{k=1}^{N}\bar{l}\|\boldsymbol{z}_k(t)\|=\\ &\sum_{k=2}^{N}\bar{l}\|\boldsymbol{z}_k(t)\|,\end{aligned} \tag{14}$$

where u_{ik} is the k-th element of $\boldsymbol{U}_i$ and $\bar{l}=\max l_k$. Therefore, the following inequalities hold:

$$\sum_{i=2}^{N}(\|\boldsymbol{g}_i(t)\|-\bar{l}\sum_{k=2}^{N}\|\boldsymbol{z}_k(t)\|)=$$
$$\sum_{i=2}^{N}\|\boldsymbol{g}_i(t)\|-\bar{l}\sum_{i=2}^{N}\sum_{k=2}^{N}\|\boldsymbol{z}_k(t)\|=$$
$$\sum_{i=2}^{N}(\|\boldsymbol{g}_i(t)\|-(N-1)\bar{l}\|\boldsymbol{z}_i(t)\|)\leqslant 0, \tag{15}$$

if the following inequalities are satisfied

$$\|\boldsymbol{g}_i(t)\|-(N-1)\bar{l}\|\boldsymbol{z}_i(t)\|\leqslant 0,\ i=2,\cdots,N. \tag{16}$$

From Eq.(11) and inequality (16), there exists a positive diagonal matrix $\boldsymbol{S}_i$, such that

$$\boldsymbol{\xi}_i^{\mathrm{T}}(t)\mathrm{diag}\{-(N-1)\bar{l}\boldsymbol{S}_i,0,0,0,\boldsymbol{S}_i\}\boldsymbol{\xi}_i(t)=$$
$$\boldsymbol{\xi}_i^{\mathrm{T}}(t)\boldsymbol{\Phi}_i\boldsymbol{\xi}_i(t)\leqslant 0. \tag{17}$$

Theorem 1 The singular error dynamical network (8) is asymptotically stable with any time-varying delays $h(t)$ if there exist positive constants α_i and matrices $\boldsymbol{P}_i>0$, $\boldsymbol{Q}_{ij}>0$, $\boldsymbol{R}_{ij}>0$, $\boldsymbol{G}_{i11}>0$, $\boldsymbol{G}_{i22}>0(j=1,2)$; positive diagonal matrix $\boldsymbol{S}_i$ and slack matrices $\boldsymbol{G}_{i12}$, $\boldsymbol{X}_{ik}$, $\boldsymbol{Y}_{ik}$, $\boldsymbol{M}_{ik}$, $\boldsymbol{N}_{ik}(k=1,2,3)$ of appropriate dimensions such that the following LMIs hold

$$\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_i=\boldsymbol{P}_i\boldsymbol{E}\geqslant 0, \tag{18}$$

$$\begin{pmatrix}\boldsymbol{G}_{i11} & \boldsymbol{G}_{i12}\\ * & \boldsymbol{G}_{i22}\end{pmatrix}>0, \tag{19}$$

$$\begin{pmatrix}\boldsymbol{\Pi}_{ik}+\Sigma_{ik}+\Sigma_{ik}^{\mathrm{T}} & \Sigma_{i12} & \Sigma_{i13}^{kj}\\ * & \Sigma_{i22} & 0\\ * & * & -\boldsymbol{R}_{ik}\end{pmatrix}<0, \tag{20}$$

$$i=2,\cdots,N,\ k=1,2,$$

$$\boldsymbol{\Pi}_{i1}=\begin{pmatrix}\boldsymbol{\Delta}_{i11} & c_2\lambda_i\boldsymbol{P}_i\boldsymbol{\Gamma}_2 & \boldsymbol{G}_{i12} & 0 & \boldsymbol{P}_i\\ * & (h_{\mathrm{d}}-1)\boldsymbol{Q}_{i1} & 0 & 0 & 0\\ * & * & \boldsymbol{\Delta}_{i33} & \boldsymbol{\Delta}_{i34} & 0\\ * & * & * & \boldsymbol{\Delta}_{i44} & 0\\ * & * & * & * & -\alpha_i\boldsymbol{S}_i\end{pmatrix},$$

$$\boldsymbol{\Pi}_{i2}=\begin{pmatrix}\bar{\boldsymbol{\Delta}}_{i11} & c_2\lambda_i\boldsymbol{P}_i\boldsymbol{\Gamma}_2 & \bar{\boldsymbol{\Delta}}_{i13} & 0 & \boldsymbol{P}_i\\ * & (h_{\mathrm{d}}-1)\boldsymbol{Q}_{i1} & 0 & 0 & 0\\ * & * & \bar{\boldsymbol{\Delta}}_{i33} & -\boldsymbol{G}_{i12} & 0\\ * & * & * & \bar{\boldsymbol{\Delta}}_{i44} & 0\\ * & * & * & * & -\alpha_i\boldsymbol{S}_i\end{pmatrix},$$

where

$$\boldsymbol{\Delta}_{i11}=(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)+\boldsymbol{Q}_{i1}+\boldsymbol{G}_{i11}+\alpha_i(N-1)\bar{l}\boldsymbol{S}_i,$$
$$\boldsymbol{\Delta}_{i33}=\boldsymbol{Q}_{i2}+\boldsymbol{G}_{i22}-\boldsymbol{G}_{i11}-\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E},$$
$$\boldsymbol{\Delta}_{i34}=-\boldsymbol{G}_{i12}+\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E},$$
$$\boldsymbol{\Delta}_{i44}=-\boldsymbol{Q}_{i2}-\boldsymbol{G}_{i22}-\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E},$$
$$\bar{\boldsymbol{\Delta}}_{i11}=(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)+\boldsymbol{Q}_{i1}+\boldsymbol{G}_{i11}+\alpha_i(N-1)\bar{l}\boldsymbol{S}_i-\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E},$$
$$\bar{\boldsymbol{\Delta}}_{i13}=\boldsymbol{G}_{i12}+\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E},$$
$$\bar{\boldsymbol{\Delta}}_{i33}=\boldsymbol{Q}_{i2}+\boldsymbol{G}_{i22}-\boldsymbol{G}_{i11}-\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E},$$
$$\bar{\boldsymbol{\Delta}}_{i44}=-\boldsymbol{Q}_{i2}-\boldsymbol{G}_{i22},$$

$$\Sigma_{i12}=\boldsymbol{\eta}_i^{\mathrm{T}}[\sqrt{\frac{h}{2}}\boldsymbol{R}_{i1}\ \sqrt{\frac{h}{2}}\boldsymbol{R}_{i2}],\ \Sigma_{i13}^{11}=\sqrt{\frac{h}{2}}\boldsymbol{X}_{ia},$$
$$\Sigma_{i13}^{12}=\sqrt{\frac{h}{2}}\boldsymbol{Y}_{ia},\ \Sigma_{i13}^{21}=\sqrt{\frac{h}{2}}\boldsymbol{M}_{ia},\ \Sigma_{i13}^{22}=\sqrt{\frac{h}{2}}\boldsymbol{N}_{ia},$$
$$\Sigma_{i22}=\mathrm{diag}\{-\boldsymbol{R}_{i1},-\boldsymbol{R}_{i2}\},$$
$$\Sigma_{i1}=[\boldsymbol{X}_{ia}\ \ \boldsymbol{Y}_{ia}-\boldsymbol{X}_{ia}\ \ -\boldsymbol{Y}_{ia}\ \ 0\ \ 0]\boldsymbol{E},$$
$$\Sigma_{i2}=[0\ \ -\boldsymbol{M}_{ia}+\boldsymbol{N}_{ia}\ \ \boldsymbol{M}_{ia}\ \ -\boldsymbol{N}_{ia}\ \ 0]\boldsymbol{E},$$
$$\boldsymbol{X}_{ia}=[\boldsymbol{X}_{i1}^{\mathrm{T}}\ \ \boldsymbol{X}_{i2}^{\mathrm{T}}\ \ \boldsymbol{X}_{i3}^{\mathrm{T}}\ \ 0\ \ 0\ \ 0]^{\mathrm{T}},$$
$$\boldsymbol{Y}_{ia}=[\boldsymbol{Y}_{i1}^{\mathrm{T}}\ \ \boldsymbol{Y}_{i2}^{\mathrm{T}}\ \ \boldsymbol{Y}_{i3}^{\mathrm{T}}\ \ 0\ \ 0\ \ 0]^{\mathrm{T}},$$
$$\boldsymbol{M}_{ia}=[0\ \ \boldsymbol{M}_{i1}^{\mathrm{T}}\ \ \boldsymbol{M}_{i2}^{\mathrm{T}}\ \ \boldsymbol{M}_{i3}^{\mathrm{T}}\ \ 0\ \ 0]^{\mathrm{T}},$$
$$\boldsymbol{N}_{ia}=[0\ \ \boldsymbol{N}_{i1}^{\mathrm{T}}\ \ \boldsymbol{N}_{i2}^{\mathrm{T}}\ \ \boldsymbol{N}_{i3}^{\mathrm{T}}\ \ 0\ \ 0]^{\mathrm{T}}.$$

Proof Construct a Lyapunov－Krasovskii functional

$$V_i(\boldsymbol{z}_i(t))=V_{i1}(\boldsymbol{z}_i(t))+V_{i2}(\boldsymbol{z}_i(t))+V_{i3}(\boldsymbol{z}_i(t)), \tag{21}$$

where

$$V_{i1}(\boldsymbol{z}_i(t))=\boldsymbol{z}_i^{\mathrm{T}}(t)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{z}_i(t)+\int_{t-h(t)}^{t}\boldsymbol{z}_i^{\mathrm{T}}(s)\boldsymbol{Q}_{i1}\boldsymbol{z}_i(s)\mathrm{d}s+\int_{t-h}^{t-\frac{h}{2}}\boldsymbol{z}_i^{\mathrm{T}}(s)\boldsymbol{Q}_{i2}\boldsymbol{z}_i(s)\mathrm{d}s,$$

$$V_{i2}(\boldsymbol{z}_i(t))=\int_{t-\frac{h}{2}}^{t}\begin{pmatrix}\boldsymbol{z}_i(s)\\ \boldsymbol{z}_i(s-\frac{h}{2})\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{G}_{i11} & \boldsymbol{G}_{i12}\\ * & \boldsymbol{G}_{i22}\end{pmatrix}\begin{pmatrix}\boldsymbol{z}_i(s)\\ \boldsymbol{z}_i(s-\frac{h}{2})\end{pmatrix}\mathrm{d}s,$$

$$V_{i3}(\boldsymbol{z}_i(t))=\int_{t-\frac{h}{2}}^{t}\int_{\theta}^{t}\dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}\theta\mathrm{d}s+\int_{t-h}^{t-\frac{h}{2}}\int_{\theta}^{t}\dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}\theta\mathrm{d}s.$$

The time derivative of $V_{i1}(\boldsymbol{z}_i(t))$ with respect to time along the trajectory of Eq.(8) is as follows:

$$\begin{aligned}\dot{V}_{i1}=&\\ &\boldsymbol{z}_i^{\mathrm{T}}(t)(\boldsymbol{P}_i(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)+(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\Gamma}_1)^{\mathrm{T}}\boldsymbol{P}_i)\boldsymbol{z}_i(t)+\\ &2\boldsymbol{z}_i^{\mathrm{T}}(t)c_2\lambda_i\boldsymbol{\Gamma}_2\boldsymbol{z}_i(t-h(t))+2\boldsymbol{z}_i^{\mathrm{T}}(t)\boldsymbol{P}_i\boldsymbol{g}_i(t)+\\ &\boldsymbol{z}_i^{\mathrm{T}}(t)\boldsymbol{Q}_{i1}\boldsymbol{z}_i(t)-(1-h_{\mathrm{d}})\boldsymbol{z}_i^{\mathrm{T}}(t-h(t))\boldsymbol{Q}_{i1}\times\\ &\boldsymbol{z}^{\mathrm{T}}(t-h(t))+\boldsymbol{z}_i^{\mathrm{T}}(t-\frac{h}{2})\boldsymbol{Q}_{i2}\boldsymbol{z}^{\mathrm{T}}(t-\frac{h}{2})-\\ &\boldsymbol{z}_i^{\mathrm{T}}(t-h))\boldsymbol{Q}_{i2}\boldsymbol{z}^{\mathrm{T}}(t-h).\end{aligned} \tag{22}$$

While the time derivative of $V_{i2}(\boldsymbol{z}_i(t))$ and $V_{i3}(\boldsymbol{z}_i(t))$ are as follows:

$$\begin{aligned}\dot{V}_{i2}=&\begin{pmatrix}\boldsymbol{z}_i(t)\\ \boldsymbol{z}_i(t-\frac{h}{2})\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{G}_{i11} & \boldsymbol{G}_{i12}\\ * & \boldsymbol{G}_{i22}\end{pmatrix}\begin{pmatrix}\boldsymbol{z}_i(t)\\ \boldsymbol{z}_i(t-\frac{h}{2})\end{pmatrix}-\\ &\begin{pmatrix}\boldsymbol{z}_i(t-\frac{h}{2})\\ \boldsymbol{z}_i(t-h)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{G}_{i11} & \boldsymbol{G}_{i12}\\ * & \boldsymbol{G}_{i22}\end{pmatrix}\begin{pmatrix}\boldsymbol{z}_i(t-\frac{h}{2})\\ \boldsymbol{z}_i(t-h)\end{pmatrix},\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{V}_{i3}=&\dot{\boldsymbol{z}}_i^{\mathrm{T}}(t)\frac{h}{2}\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{R}_{i1}+\boldsymbol{R}_{i2})\boldsymbol{E}\dot{\boldsymbol{z}}_i(t)-\\ &\int_{t-\frac{h}{2}}^{t}\dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s-\\ &\int_{t-h}^{t-\frac{h}{2}}\dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s.\end{aligned} \tag{24}$$

Now, for any $t>0$, $h(t)\in[0,\frac{h}{2}]$, or $h(t)\in[\frac{h}{2},h]$, define $\Delta_1=\{t:h(t)\in[0,\frac{h}{2}]\}$, $\Delta_2=\{t:h(t)\in[\frac{h}{2},h]\}$. In the following, we will discuss the variation of $\dot{V}_{i3}$ for two cases:

Case 1 For $t \in \varDelta_1$, by using Lemma 3 and Lemma 4 we have that

$$-\int_{t-h}^{t-\frac{h}{2}} \dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s \leqslant \frac{2}{h}\begin{pmatrix}\boldsymbol{z}_i(t-\frac{h}{2})\\ \boldsymbol{z}_i(t-h)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}-\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\\ * & -\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\end{pmatrix}\cdot \begin{pmatrix}\boldsymbol{z}_i(t-\frac{h}{2})\\ \boldsymbol{z}_i(t-h)\end{pmatrix}, \tag{25}$$

$$-\int_{t-\frac{h}{2}}^{t} \dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s \leqslant \boldsymbol{\zeta}_{i1}^{\mathrm{T}}(t)(h(t)\boldsymbol{X}_i\boldsymbol{R}_{i1}^{-1}\boldsymbol{X}_i^{\mathrm{T}} + (\frac{h}{2}-h(t))\boldsymbol{Y}_i\boldsymbol{R}_{i1}^{-1}\boldsymbol{Y}_i^{\mathrm{T}} + [\boldsymbol{X}_i\ \ \boldsymbol{Y}_i-\boldsymbol{X}_i\ \ -\boldsymbol{Y}_i]\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}[\boldsymbol{X}_i\ \ \boldsymbol{Y}_i-\boldsymbol{X}_i\ \ -\boldsymbol{Y}_i]^{\mathrm{T}})\boldsymbol{\zeta}_{i1}(t), \tag{26}$$

where

$$\boldsymbol{\zeta}_{i1}(t) = (\boldsymbol{z}_i^{\mathrm{T}}(t),\ \boldsymbol{z}_i^{\mathrm{T}}(t-h(t)),\ \boldsymbol{z}_i^{\mathrm{T}}(t-\frac{h}{2})),$$
$$\boldsymbol{X}_i = [\boldsymbol{X}_{i1}^{\mathrm{T}}\ \ \boldsymbol{X}_{i2}^{\mathrm{T}}\ \ \boldsymbol{X}_{i3}^{\mathrm{T}}],\ \boldsymbol{Y}_i = [\boldsymbol{Y}_{i1}^{\mathrm{T}}\ \ \boldsymbol{Y}_{i2}^{\mathrm{T}}\ \ \boldsymbol{Y}_{i3}^{\mathrm{T}}].$$

From Eqs.(17)(24)–(26), using Lemma 6, it follows that

$$\dot{V}_i \leqslant \dot{V}_{i1} + \dot{V}_{i2} + \dot{V}_{i3} - \alpha_i\boldsymbol{\xi}_i^{\mathrm{T}}(t)\boldsymbol{\varPhi}_i\boldsymbol{\xi}_i(t) \leqslant \boldsymbol{\xi}_i^{\mathrm{T}}(t)(\boldsymbol{\varPi}_{i1} + \varSigma_{i1} + \varSigma_{i1}^{\mathrm{T}} + \boldsymbol{\eta}_i^{\mathrm{T}}\frac{h}{2}(\boldsymbol{R}_{i1}+\boldsymbol{R}_{i2})\boldsymbol{\eta}_i + h(t)\boldsymbol{X}_{ia}\boldsymbol{R}_{i1}^{-1}\boldsymbol{X}_{ia}^{\mathrm{T}} + (\frac{h}{2}-h(t))\boldsymbol{Y}_{ia}\boldsymbol{R}_{i1}^{-1}\boldsymbol{Y}_{ia}^{\mathrm{T}})\boldsymbol{\xi}_i(t). \tag{27}$$

From Eqs.(18)–(21), when $k=1$; $j=1$ and $j=2$, using Schur complement, we have that

$$\boldsymbol{\varPi}_{i1} + \varSigma_{i1} + \varSigma_{i1}^{\mathrm{T}} + \boldsymbol{\eta}_i^{\mathrm{T}}\frac{h}{2}(\boldsymbol{R}_{i1}+\boldsymbol{R}_{i2})\boldsymbol{\eta}_i + \frac{h}{2}\boldsymbol{X}_{ia}\boldsymbol{R}_{i1}^{-1}\boldsymbol{X}_{ia}^{\mathrm{T}} < 0,$$
$$\boldsymbol{\varPi}_{i1} + \varSigma_{i1} + \varSigma_{i1}^{\mathrm{T}} + \boldsymbol{\eta}_i^{\mathrm{T}}\frac{h}{2}(\boldsymbol{R}_{i1}+\boldsymbol{R}_{i2})\boldsymbol{\eta}_i + \frac{h}{2}\boldsymbol{Y}_{ia}\boldsymbol{R}_{i1}^{-1}\boldsymbol{Y}_{ia}^{\mathrm{T}} < 0.$$

Using Lemma 5, one gets that $\dot{\boldsymbol{V}}(\boldsymbol{z}_i(t)) < 0$.

Case 2 For $t \in \varDelta_1$, following a similar line of arguments as that in Case 1, we have

$$-\int_{t-\frac{h}{2}}^{t} \dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s \leqslant \frac{2}{h}\begin{pmatrix}\boldsymbol{z}_i(t)\\ \boldsymbol{z}_i(t-\frac{h}{2})\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}-\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\\ * & -\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\end{pmatrix}\begin{pmatrix}\boldsymbol{z}_i(t)\\ \boldsymbol{z}_i(t-\frac{h}{2})\end{pmatrix}, \tag{28}$$

$$-\int_{t-h}^{t-\frac{h}{2}} \dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s \leqslant \boldsymbol{\zeta}_{i2}^{\mathrm{T}}(t)((h(t)-\frac{h}{2})\boldsymbol{M}_i\boldsymbol{R}_{i2}^{-1}\boldsymbol{M}_i^{\mathrm{T}} + (h-h(t))\boldsymbol{N}_i\boldsymbol{R}_{i2}^{-1}\boldsymbol{N}_i^{\mathrm{T}} + (-\boldsymbol{M}_i + \boldsymbol{N}_i\boldsymbol{M}_i - \boldsymbol{N}_i)\boldsymbol{E} + \boldsymbol{E}^{\mathrm{T}}(-\boldsymbol{M}_i + \boldsymbol{N}_i\boldsymbol{M}_i - \boldsymbol{N}_i)^{\mathrm{T}})\boldsymbol{\zeta}_{i2}(t), \tag{29}$$

where $\boldsymbol{\zeta}_{i2}(t) = (\boldsymbol{z}_i^{\mathrm{T}}(t-h(t)),\ \boldsymbol{z}_i^{\mathrm{T}}(t-\frac{h}{2}),\ \boldsymbol{z}_i^{\mathrm{T}}(t-h))$, $\boldsymbol{M}_i = [\boldsymbol{M}_{i1}^{\mathrm{T}}\ \ \boldsymbol{M}_{i2}^{\mathrm{T}}\ \ \boldsymbol{M}_{i3}^{\mathrm{T}}]$, $\boldsymbol{N}_i = [\boldsymbol{N}_{i1}^{\mathrm{T}}\ \ \boldsymbol{N}_{i2}^{\mathrm{T}}\ \ \boldsymbol{N}_{i3}^{\mathrm{T}}]$.

From Eqs.(17)(24)(28)–(29), using Lemma 6, it follows that

$$\dot{V}_i = \dot{V}_{i1} + \dot{V}_{i2} + \dot{V}_{i3} - \alpha_i\boldsymbol{\xi}_i^{\mathrm{T}}(t)\boldsymbol{\varPhi}_i\boldsymbol{\xi}_i(t) \leqslant \boldsymbol{\xi}_i^{\mathrm{T}}(t)(\boldsymbol{\varPi}_{i2} + \varSigma_{i2} + \varSigma_{i2}^{\mathrm{T}} + \boldsymbol{\eta}_i^{\mathrm{T}}\frac{h}{2}(\boldsymbol{R}_{i1}+\boldsymbol{R}_{i2})\boldsymbol{\eta}_i + (h(t)-\frac{h}{2})\boldsymbol{M}_{ia}\boldsymbol{R}_{i2}^{-1}\boldsymbol{M}_{ia}^{\mathrm{T}} + (h-h(t))\boldsymbol{N}_{ia}\boldsymbol{R}_{i2}^{-1}\boldsymbol{N}_{ia}^{\mathrm{T}})\boldsymbol{\xi}_i(t). \tag{30}$$

From inequalities.(18)–(21), when $k=2$, $j=1$ and $j=2$, using Schur complement, we have that

$$\boldsymbol{\varPi}_{i1} + \varSigma_{i2} + \varSigma_{i2}^{\mathrm{T}} + \boldsymbol{\eta}_i^{\mathrm{T}}\frac{h}{2}(\boldsymbol{R}_{i1} + \boldsymbol{R}_{i2})\boldsymbol{\eta}_i + \frac{h}{2}\boldsymbol{M}_{ia}\boldsymbol{R}_{i2}^{-1}\boldsymbol{M}_{ia}^{\mathrm{T}} < 0,$$
$$\boldsymbol{\varPi}_{i1} + \varSigma_{i2} + \varSigma_{i2}^{\mathrm{T}} + \boldsymbol{\eta}_i^{\mathrm{T}}\frac{h}{2}(\boldsymbol{R}_{i1} + \boldsymbol{R}_{i2})\boldsymbol{\eta}_i + \frac{h}{2}\boldsymbol{N}_{ia}\boldsymbol{R}_{i2}^{-1}\boldsymbol{N}_{ia}^{\mathrm{T}} < 0.$$

Using Lemma 5, one gets that $\dot{\boldsymbol{V}}(\boldsymbol{z}_i(t)) < 0$.

Note that $\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_i = \boldsymbol{P}_i\boldsymbol{E} \geqslant 0$, one cannot obtain the stable result via the Lyapunov stability theory because the rank of $\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_i$ in the Lyapunov function $\boldsymbol{V}_{i1}(\boldsymbol{z}_i)$ is $r < n$. According to Lemma 2, it is obvious that the pair $(\boldsymbol{E}, \boldsymbol{A} + c_1\lambda_i\boldsymbol{\varGamma}_1)$ is regular and impulse free if the inequalities (18)–(21) hold[20,30]. Then, there exist matrices $\boldsymbol{H}_{i1} \in \mathbb{R}^{r\times n}$, $\boldsymbol{H}_{i2} \in \mathbb{R}^{(n-r)\times n}$, $\boldsymbol{K}_{i1} \in \mathbb{R}^{n\times r}$, $\boldsymbol{K}_{i2} \in \mathbb{R}^{n\times(n-r)}$, such that $\boldsymbol{H}_i = [\boldsymbol{H}_{i1}^{\mathrm{T}}\ \ \boldsymbol{H}_{i2}^{\mathrm{T}}]$ and $\boldsymbol{K}_i = [\boldsymbol{K}_{i1}^{\mathrm{T}}\ \ \boldsymbol{K}_{i2}^{\mathrm{T}}]^{\mathrm{T}}$ are two nonsingular matrices and the following standard decomposition holds:

$$\boldsymbol{H}_i\boldsymbol{E}\boldsymbol{K}_i = \mathrm{diag}\{\boldsymbol{I}_r, 0\},$$
$$\boldsymbol{H}_i(\boldsymbol{A}+c_1\lambda_i\boldsymbol{\varGamma}_1)\boldsymbol{K}_i = \mathrm{diag}\{\bar{\boldsymbol{A}}_i, \boldsymbol{I}_{n-r}\}, \tag{31}$$

where $\bar{\boldsymbol{A}}_i \in \mathbb{R}^{r\times r}$, $i = 2, \cdots, N$. The network system (8) is equivalent to

$$\begin{cases}\dot{\boldsymbol{z}}_i^{(1)} = \bar{\boldsymbol{A}}_i\boldsymbol{z}_i^{(1)} + \boldsymbol{H}_{i1}\boldsymbol{g}_i + c_2\lambda_i\boldsymbol{H}_{i1}\boldsymbol{\varGamma}_{2r}\boldsymbol{K}_{i1}\boldsymbol{z}_i^{(1)}(t-h(t)),\\ 0 = \boldsymbol{z}_i^{(2)} + \boldsymbol{H}_{i2}\boldsymbol{g}_i + c_2\lambda_i\boldsymbol{H}_{i2}\boldsymbol{\varGamma}_{2(n-r)}\boldsymbol{K}_{i2}\boldsymbol{z}_i^{(2)}(t-h(t)),\\ i = 2, \cdots, N,\end{cases} \tag{32}$$

where $\boldsymbol{y_i}(t) = \boldsymbol{K}_i^{-1}\boldsymbol{z}_i(t) = \begin{pmatrix}\boldsymbol{z}_i^{(1)}(t)\\ \boldsymbol{z}_i^{(2)}(t)\end{pmatrix}$, $\boldsymbol{\varGamma}_{2r} = \mathrm{diag}\{\tau_1, \cdots, \tau_r\}$ and $\boldsymbol{\varGamma}_{2(n-r)} = \mathrm{diag}\{\tau_{r+1}, \cdots, \tau_n\}$.

Let $\boldsymbol{H}_i^{-\mathrm{T}}\boldsymbol{P}_i\boldsymbol{K}_i = \begin{pmatrix}\boldsymbol{P}_i^{(1)} & \boldsymbol{P}_i^{(2)}\\ \boldsymbol{P}_i^{(3)} & \boldsymbol{P}_i^{(4)}\end{pmatrix}$. Then according to Eqs.(18) and (31), it is easy to see that $\boldsymbol{P}_i^{(1)} = \boldsymbol{P}_i^{(1)\mathrm{T}}$ and $\boldsymbol{P}_i^{(2)} = 0$[28]. Hence,

$$V_{i1}(\boldsymbol{z}_i(t)) = (\boldsymbol{z}_i^{(1)}(t))^{\mathrm{T}}\boldsymbol{P}_i^{(1)}(\boldsymbol{z}_i^{(1)}(t)) + \int_{t-h(t)}^{t} \boldsymbol{z}_i^{\mathrm{T}}(s)\boldsymbol{Q}_{i1}\boldsymbol{z}_i(s)\mathrm{d}s + \int_{t-h}^{t-\frac{h}{2}} \boldsymbol{z}_i^{\mathrm{T}}(s)\boldsymbol{Q}_{i2}\boldsymbol{z}_i(s)\mathrm{d}s. \tag{33}$$

From $\dot{V}_i(\boldsymbol{z}_i) < 0$, $\boldsymbol{z}_i^{(1)}(t)$ of system (8) is asymptotically stable, i.e., $\lim\limits_{t\to\infty}\|\boldsymbol{z}_i^{(1)}(t)\|=0$, $i=2,\cdots,N$. In the following, we show that $\boldsymbol{z}_i^{(2)}(t)$ are also asymptotically stable. From Eqs.(32) and similar with [30], choosing $\boldsymbol{H}_{i2}$ such that $\boldsymbol{H}_{i2}\boldsymbol{H}_{i2}^{\mathrm{T}}=\boldsymbol{I}_{n-r}$ which implies that $\|\boldsymbol{H}_{i2}\|=1$ and using Lemma 1, we have

$$\begin{aligned}&\|\boldsymbol{z}_i^{(2)}\|=\\&\|\boldsymbol{H}_{i2}\boldsymbol{g}_i+c_2\lambda_i\boldsymbol{H}_{i2}\boldsymbol{\varGamma}_{2(n-r)}\boldsymbol{K}_{i2}\boldsymbol{z}_i^{(2)}(t-h(t))\|\leqslant\\&\|\boldsymbol{H}_{i2}\boldsymbol{g}_i\|+\|c_2\lambda_i\boldsymbol{H}_{i2}\boldsymbol{\varGamma}_{2(n-r)}\boldsymbol{K}_{i2}\boldsymbol{z}_i^{(2)}(t-h(t))\|\leqslant\\&\|\boldsymbol{H}_{i2}\|\|\boldsymbol{g}_i\|+c_2\max(\lambda_i)\|\boldsymbol{H}_{i2}\|\|\boldsymbol{K}_{i2}\|\|\boldsymbol{\varGamma}_{2(n-r)}\|\cdot\\&\|\boldsymbol{z}_i^{(2)}(t-h(t))\|\leqslant\sum_{i=2}^{N}\bar{l}\|\boldsymbol{K}_i\|(\|\boldsymbol{z}_i^{(1)}\|+\|\boldsymbol{z}_i^{(2)}\|).\end{aligned}\tag{34}$$

Then, $\sum\limits_{i=2}^{N}\|\boldsymbol{z}_i^{(2)}\|\leqslant(N-1)\sum\limits_{i=2}^{N}\bar{l}\|\boldsymbol{K}_i\|(\|\boldsymbol{z}_i^{(1)}\|+\|\boldsymbol{z}_i^{(2)}\|)$, i.e., $\sum\limits_{i=2}^{N}(1-(N-1)\bar{l}\|\boldsymbol{K}_i\|)\|\boldsymbol{z}_i^{(2)}\|\leqslant(N-1)\bar{l}\sum\limits_{i=2}^{N}\|\boldsymbol{K}_i\|\|\boldsymbol{z}_i^{(1)}\|$. Therefore, one can obtain $\lim\limits_{t\to\infty}\|\boldsymbol{z}_i^{(2)}(t)\|=0$, $i=2,\cdots,N$, if we choose $\boldsymbol{K}_i$ such that $1-(N-1)\bar{l}\|\boldsymbol{K}_i\|>0$. This completes the proof.

Remark 3 In this paper, by using the delay decomposition method and for convenience, the delay interval $[0,h]$ is divided into two equivalent subintervals $[0,\frac{h}{2}]$ and $[\frac{h}{2},h]$. And in estimating an upper bound of $\dot{V}_{i3}(\boldsymbol{z}_i(t))$, the term $-\int_{t-\frac{h}{2}}^{t}\dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s$ and the term $-\int_{t-h}^{t-\frac{h}{2}}\dot{\boldsymbol{z}}_i^{\mathrm{T}}(s)\cdot\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i2}\boldsymbol{E}\dot{\boldsymbol{z}}_i(s)\mathrm{d}s$ are computed in two cases respectively, where different free-weighting matrix variables are fully used at each cases. These can give an improved feasible region for delay-dependent stability criterion. In fact, to further reduce the conservation, we should divide the delay interval $[0,h]$ into $N(N\geqslant3)$ parts for generalization[25,39].

Remark 4 Recently, the synchronization of singular complex dynamical networks with coupling delays is investigated in [34–35]. It is obvious that in our paper, the synchronization problem reduces to that in [34–35] when $\boldsymbol{\varGamma}_1=0$, i.e., it is assumed that there exists the information communication of nodes only by the edges at time $t-h(t)$. However, some examples in Section 4 show that our results are less conservative than the previous result.

Remark 5 When $\boldsymbol{\varGamma}_1=0$, the system (8) reduces to the following corresponding system[34–35]:

$$\begin{aligned}\boldsymbol{E}\dot{\boldsymbol{z}}_i(t)=&\ \boldsymbol{A}\boldsymbol{z}_i(t)+\boldsymbol{g}_i(t)+c_2\lambda_i\boldsymbol{\varGamma}_2\boldsymbol{z}_i(t-h(t)),\\&i=2,\cdots,N.\end{aligned}\tag{35}$$

Then similar to the proof of Theorem 1, the following much less conservative synchronization criterion can be derived:

Corollary 1 The singular error dynamical network (35) is asymptotically stable with any time-varying delays $h(t)$ if there exist positive constants α_i and matrices $\boldsymbol{P}_i>0$, $\boldsymbol{Q}_{ij}>0$, $\boldsymbol{R}_{ij}>0$, $\boldsymbol{G}_{i11}>0$, $\boldsymbol{G}_{i22}>0$, $(j=1,2)$; positive diagonal matrix $\boldsymbol{S}_i$ and slack matrices $\boldsymbol{G}_{i12}$, $\boldsymbol{X}_{ik}$, $\boldsymbol{Y}_{ik}$, $\boldsymbol{M}_{ik}$, $\boldsymbol{N}_{ik}$, $k=1,2,3$ of appropriate dimensions such that the following LMIs hold:

$$\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_i=\boldsymbol{P}_i\boldsymbol{E}\geqslant0,\tag{36}$$

$$\begin{pmatrix}\boldsymbol{G}_{i11}&\boldsymbol{G}_{i12}\\ *&\boldsymbol{G}_{i22}\end{pmatrix}>0,\tag{37}$$

$$\begin{aligned}&\begin{pmatrix}\tilde{\boldsymbol{\varPi}}_{ik}+\varSigma_{ik}+\varSigma_{ik}^{\mathrm{T}}&\varSigma_{i12}&\varSigma_{i13}^{kj}\\ *&\varSigma_{i22}&0\\ *&*&-\boldsymbol{R}_{ik}\end{pmatrix}<0,\\&i=2,\cdots,N;\ k=1,2.\end{aligned}\tag{38}$$

Here, $\tilde{\boldsymbol{\varPi}}_{i1}$ and $\tilde{\boldsymbol{\varPi}}_{i2}$ are defined as: replacing $\boldsymbol{\Delta}_{i11}$, $\bar{\boldsymbol{\Delta}}_{i11}$ in $\boldsymbol{\varPi}_{i1}$ and $\boldsymbol{\varPi}_{i2}$ of Theorem 1, respectively by: $\tilde{\boldsymbol{\Delta}}_{i11}=\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}+\boldsymbol{Q}_{i1}+\boldsymbol{G}_{i11}+\alpha_i(N-1)\bar{l}\boldsymbol{S}_i$, $\tilde{\bar{\boldsymbol{\Delta}}}_{i11}=\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{A}+\boldsymbol{Q}_{i1}+\boldsymbol{G}_{i11}+\alpha_i(N-1)\bar{l}\boldsymbol{S}_i-\frac{2}{h}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_{i1}\boldsymbol{E}$.

4 Numerical examples

In this section, some numerical examples are used to illustrate the effectiveness of the proposed synchronization criteria given in this paper.

Example 1 Consider the following time-varying delayed singular complex network system

$$\begin{aligned}&\boldsymbol{E}\dot{\boldsymbol{x}}_i(t)=\\&\boldsymbol{A}\boldsymbol{x}_i(t)+\boldsymbol{f}(\boldsymbol{x}_i(t),t)+c_1\sum_{j=1}^{6}g_{ij}\boldsymbol{\varGamma}_1\boldsymbol{x}_j(t)+\\&c_2\sum_{j=1}^{6}g_{ij}\boldsymbol{\varGamma}_2\boldsymbol{x}_j(t-h(t)),\ t>0,\ i=1,\cdots,6.\end{aligned}\tag{39}$$

Here,

$$\boldsymbol{E}=\begin{pmatrix}8&0\\0&0\end{pmatrix},\ \boldsymbol{A}=\begin{pmatrix}-2&1\\1&-2\end{pmatrix},\ \boldsymbol{\varGamma}_1=\begin{pmatrix}\theta_1&0\\0&\theta_2\end{pmatrix},$$

$$\boldsymbol{\varGamma}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \boldsymbol{x}_i(t)=(\boldsymbol{x}_{i1}^{\mathrm{T}}(t),\boldsymbol{x}_{i2}^{\mathrm{T}}(t))^{\mathrm{T}},$$

$$\boldsymbol{G}=\begin{pmatrix}-5&1&1&1&1&1\\1&-4&1&1&1&0\\1&1&-4&1&0&1\\1&1&1&-4&1&0\\1&1&0&1&-4&1\\1&0&1&0&1&-3\end{pmatrix},$$

and $\boldsymbol{f}(\boldsymbol{x}_i(t),t)=\frac{1}{15}(\tanh(x_{i1}(t),t),\tanh(x_{i2}(t),t))^{\mathrm{T}}$. It is obvious that $\bar{l}=\frac{1}{15}$, $\boldsymbol{G}$ is an irreducible symmetric matrix, and the eigenvalues of $\boldsymbol{G}$ are $\lambda_1=0$, $\lambda_2=-3$, $\lambda_3=-4$, $\lambda_4=-5$, $\lambda_5=\lambda_6=-6$.

If we set $\boldsymbol{\varGamma}_1=\mathbf{0}$, then the system reduces to that in [33–35]. And using Corollary 1, we can compute the corresponding maximum allowable delay bounds (MADBs) h for different c_2 and h_{d}. In Table 1, the MADBs h with different c_2 and h_{d} by using Corollary 1 and method in [35] are compared. From Table 1, one can see that the proposed method presented in this paper provides less conservative result than the previous result when $\boldsymbol{\varGamma}_1=0$.

If we set $\boldsymbol{\varGamma}_1\neq0$, for simplicity, let $\boldsymbol{\varGamma}_1=0.1\boldsymbol{\varGamma}_2$, $c_1=c_2=c$, then by Theorem 1, the corresponding results are listed in Table 2. Fig.1 and Fig.2 depict the errors state response of $\boldsymbol{z}_{i1}(t)$ for the random initial conditions with $\boldsymbol{\varGamma}_1=0$, $c_2=0.1$, $h=19.905$ and $\boldsymbol{\varGamma}_1=0.1\boldsymbol{\varGamma}_2$, $c=0.1$, $h=20.339$, respectively. We can see that the synchronization errors converge to zero.

Table 1 MADBs h for $\boldsymbol{\Gamma}_1 = 0$ in Example 1

c_2	Methods	h_d		
		0	0.1	0.2
0.1	[35]	15.020	13.766	12.594
	Corollary 1 here	19.905	18.282	16.902
0.2	[35]	2.708	2.038	1.139
	Corollary 1 here	3.796	2.888	1.655

Table 2 MADBs h for $\boldsymbol{\Gamma}_1 = 0.1\boldsymbol{\Gamma}_2$ in Example 1

c	Methods	h_d		
		0	0.1	0.2
0.1	Theorem 1	20.339	17.825	15.715
0.2	Theorem 1	3.178	2.383	1.343

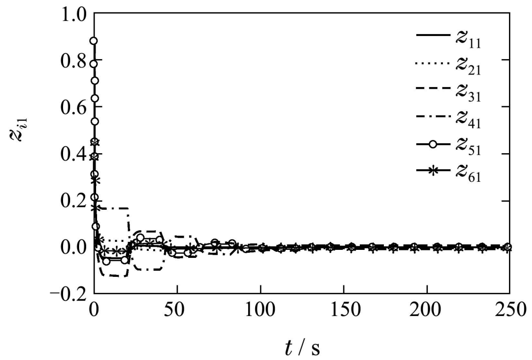

Fig. 1 The errors state response of $\boldsymbol{z}_{i1}(t)$ of Eq.(A1) with $\boldsymbol{\Gamma}_1 = 0$, $c_2 = 0.1$, $h = 19.905$

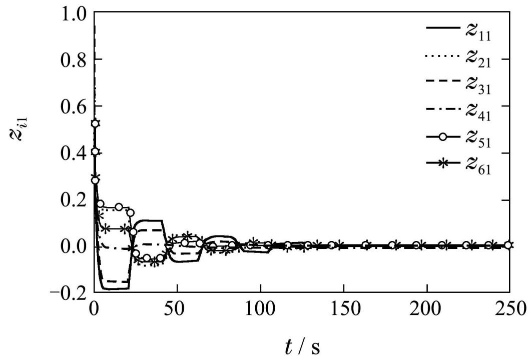

Fig. 2 The errors state response of $\boldsymbol{z}_{i1}(t)$ of Eq.(A1) with $\boldsymbol{\Gamma}_1 = 0.1\boldsymbol{\Gamma}_2$, $c = 0.1$, $h = 20.339$

Example 2 Consider the following time-varying delayed singular complex network system:

$$\boldsymbol{E}\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + c_1 \sum_{j=1}^{5} g_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t) + c_2 \sum_{j=1}^{5} g_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-h(t)),\ t>0,\quad i = 1,\cdots,5, \tag{40}$$

where

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix},\ \boldsymbol{\Gamma}_1 = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix},$$

$$\boldsymbol{\Gamma}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \boldsymbol{x}_i(t) = (\boldsymbol{x}_{i1}^\mathrm{T}(t), \boldsymbol{x}_{i2}^\mathrm{T}(t), \boldsymbol{x}_{i3}^\mathrm{T}(t))^\mathrm{T},$$

$$\boldsymbol{G} = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Similar to Example 1, if we set $\boldsymbol{\Gamma}_1 = 0$, then the system reduces to that in [9, 20–21, 35]. We can compute the corresponding MADBs of h for different c_2 and h_d. The MADBs of h with different c_2 and h_d by using Corollary 1 and method in [35] are compared in Table 3. One can see that the proposed method presented in this paper provides less conservative result than the previous result. If we set $\boldsymbol{\Gamma}_1 \neq 0$, for simplicity, $\boldsymbol{\Gamma}_1 = 0.1\boldsymbol{\Gamma}_2$, $c_1 = c_2 = c$, then by Theorem 1, the corresponding results are listed in Table 4. Fig.3 depicts the errors state response of $\boldsymbol{z}_{i1}(t)$ for the random initial conditions with $\boldsymbol{\Gamma}_1 = 0$, $c_2 = 0.3$, $h = 2.214$. We can see that the synchronization errors converge to zero under the conditions.

Table 3 MADBs h for $\boldsymbol{\Gamma}_1 = 0$ in Example 2

c_2	Methods	h_d		
		0	0.1	0.2
0.3	[35]	2.066	1.894	1.740
	Corollary 1 here	2.214	2.020	1.849
0.4	[35]	1.191	1.130	1.071
	Corollary 1 here	1.257	1.188	1.122
0.5	[35]	0.852	0.817	0.782
	Corollary 1 here	0.892	0.853	0.814
0.6	[35]	0.666	0.642	0.618
	Corollary 1 here	0.695	0.668	0.641

Table 4 MADBs h for $\boldsymbol{\Gamma}_1 = 0.1\boldsymbol{\Gamma}_2$ in Example 2

c	Methods	h_d		
		0	0.1	0.2
0.3	Theorem 1 here	2.786	2.436	2.156
0.4	Theorem 1 here	1.430	1.333	1.242
0.5	Theorem 1 here	0.988	0.836	0.886
0.6	Theorem 1 here	0.760	0.721	0.691

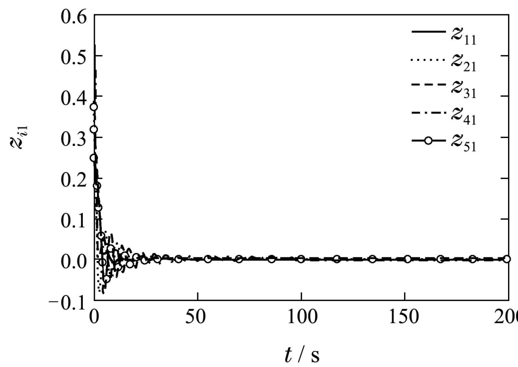

Fig. 3 The errors state response of $\boldsymbol{z}_{i1}(t)$ of Eq.(A2) with $\boldsymbol{\Gamma}_1 = 0$, $c_2 = 0.3$, $h = 2.214$

It is important to note that the obtained maximum delay bound $h = 2.214$ by Corollary 1 is very close to the true value of the maximum delay bound beyond which the synchronized states is not asymptotically stable. To show this, we assume the time-delay in the network to be $h = 2.49$. Fig.4 depicts the errors state response of $z_{i1}(t)$ for the random initial conditions with $\Gamma_1 = 0$, $c_2 = 0.3$, $h = 2.49$. We can see that the errors between the synchronized states do not converge to zero under the above conditions. Fig.5 depicts the errors state response of $z_{i1}(t)$ for the random initial conditions with $\Gamma_1 = 0.1\Gamma_2$, $c = 0.3$, $h = 2.786$. We can see that the synchronization errors converge to zero.

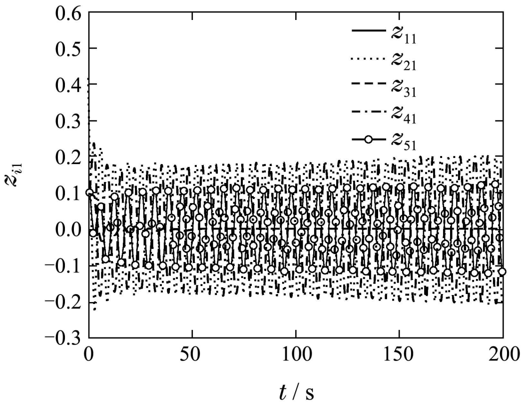

Fig. 4 The errors state response of $z_{i1}(t)$ of Eq.(A2) with $\Gamma_1 = 0$, $c_2 = 0.3$, $h = 2.49$

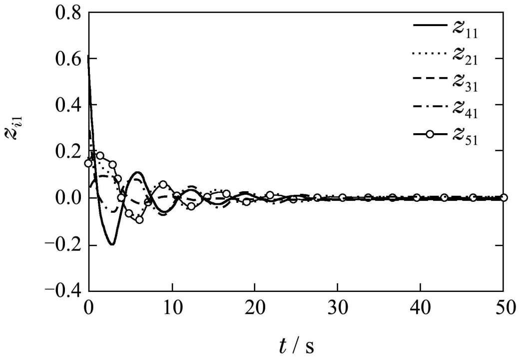

Fig. 5 The errors state response of $z_{i1}(t)$ of Eq.(A2) with $\Gamma_1 = 0.1\Gamma_2$, $c = 0.3$, $h = 2.786$

5 Conclusions

In this paper, some new synchronization stability criteria are proposed for singular complex dynamical networks with non-delayed and delayed coupling. The delay-dependent synchronization criteria are derived in the form of linear matrix inequalities. With applying some effective techniques, the proposed criteria are less conservative than the existing results. Numerical examples are used to illustrate the effectiveness of the proposed criteria and their improvements over the existent methods.

There are still a number of related interesting problems deserving further investigation. For instance, it is desirable to study synchronization problem for singular complex dynamical networks with stochastic disturbances, uncertainties, sampled data, switching topology, and so on, some of which will be investigated in the near future.

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Appendix Proof of Lemma 4

For any $\boldsymbol{R}=\boldsymbol{R}^{\mathrm{T}}>0$, the following inequality holds:

$$\int_{t-h(t)}^{t-h_1}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\zeta}(t)+\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s))^{\mathrm{T}}\boldsymbol{R}^{-1}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\zeta}(t)+\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s))\mathrm{d}s\geqslant 0, \tag{A1}$$

$$\int_{t-h_2}^{t-h(t)}(\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{\zeta}(t)+\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s))^{\mathrm{T}}\boldsymbol{R}^{-1}(\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{\zeta}(t)+\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s))\mathrm{d}s\geqslant 0. \tag{A2}$$

Hence

$$\begin{aligned}&-2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{X}\int_{t-h(t)}^{t-h_1}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s\leqslant\\&\boldsymbol{\zeta}^{\mathrm{T}}(t)(h(t)-h_1)\boldsymbol{X}\boldsymbol{R}^{-1}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\zeta}(t)+\\&\int_{t-h(t)}^{t-h_1}\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s,\\&-2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{Y}\int_{t-h_2}^{t-h(t)}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s\leqslant\\&\boldsymbol{\zeta}^{\mathrm{T}}(t)(h_2-h(t))\boldsymbol{Y}\boldsymbol{R}^{-1}\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{\zeta}(t)+\\&\int_{t-h_2}^{t-h(t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s.\end{aligned}$$

By Newton-Leibniz formula, we have that

$$\begin{aligned}&-\int_{t-h_2}^{t-h_1}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s=\\&-\int_{t-h(t)}^{t-h_1}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s-\\&\int_{t-h_2}^{t-h(t)}\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s+2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{X}\times\\&(\boldsymbol{E}\boldsymbol{x}(t-h_1)-\boldsymbol{E}\boldsymbol{x}(t-h(t))-\int_{t-h(t)}^{t-h_1}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s)+\\&2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{Y}(\boldsymbol{E}\boldsymbol{x}(t-h(t))-\boldsymbol{E}\boldsymbol{x}(t-h_2)-\int_{t-h_2}^{t-h(t)}\boldsymbol{E}\dot{\boldsymbol{x}}(s)\mathrm{d}s)\leqslant\\&\boldsymbol{\zeta}^{\mathrm{T}}(t)((h(t)-h_1)\boldsymbol{X}\boldsymbol{R}^{-1}\boldsymbol{X}^{\mathrm{T}}+(h_2-h(t))\boldsymbol{Y}\boldsymbol{R}^{-1}\boldsymbol{Y}^{\mathrm{T}}+\\&(\boldsymbol{X}\ \ \boldsymbol{Y}-\boldsymbol{X}\ \ -\boldsymbol{Y})\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{X}\ \ \boldsymbol{Y}-\boldsymbol{X}\ \ -\boldsymbol{Y})^{\mathrm{T}})\boldsymbol{\zeta}(t).\end{aligned} \tag{A3}$$

The proof of Lemma 4 is completed.

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