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分数阶迭代学习控制的收敛性分析

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摘要: 本文将传统的迭代学习控制时域和频域分析方法扩展到一类针对分数阶非线性系统的分数阶迭代学习控制时域分析方法. 提出了一类新的分数阶迭代学习控制框架并简化了收敛条件, 且证明了常增益情况下两类分数阶迭代学习控制收敛条件的等价性问题. 该讨论进一步引出了如下两个结果: 分数阶不确定系统的分数阶自适应迭代学习控制的可学习区域以及理想带阻型分数阶迭代学习控制的框架. 上述结果均得到了仿真验证.

关键词: 迭代学习控制; 分数阶微积分; 非线性系统; 收敛性; 自适应

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Convergence analysis of fractional-order iterative learning control

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Abstract: The classical time domain and frequency domain analysis of iterative learning control (ILC) are extended to a type of time domain analysis of fractional order iterative learning control (FOILC) for fractional order nonlinear systems. A novel FOILC scheme is proposed, which leads to simpler convergence condition. The equivalence of the above two FOILC schemes is shown for the constant learning gain cases, which leads to two further developments: the learnable domain of an adaptive FOILC for the uncertain fractional order systems, and a desirable band-stop FOILC scheme. Several examples are provided to illustrate the presented results.

Key words: iterative learning control; fractional calculus; nonlinear systems; convergence; adaptiveness

1 Introduction

The formal concept of iterative learning control (ILC) was published in 1978 by Uchiyama (in Japanese) and in 1984 by Arimoto et al (in English)[1]. Some earlier works of ILC can be traced back to 1967 the US patent 3555252 ‘learning control of actuators in control systems’ and the multi-pass system analysis in 1974 by Edwards and Owens[2]. The ILC method is a batch process that operates a given objective system repeatedly on a fixed time interval so that the reference can be perfectly tracked as the operation repeats. Along with the global Lipschitz condition, the ILC schemes can be easily applied to both nonlinear and linear systems with less prior model knowledge[1,3–7]. So, both the theory and applications of ILC gained increasingly attention and have been highly developed in the past three decades[8–12]. Many interesting problems of ILC have been discussed[13–22], and all of them are linked to the ‘repetition’ of the control system, such as the batch process and periodic system uncertainties and disturbances, etc[5–7,23–24]. In simple words, an ILC scheme is introduced to improve the system control performance by using the historical data even without a complete knowledge of the system to be controlled[3,8–9,21–22].

The fractional order iterative learning control (FOILC) is relatively new in ILC, which can be traced back to 2001, when, in [25], a $D^{\alpha}-$type ILC updating law was proposed and the convergence conditions were analyzed in the frequency domain, which became a main technique in the FOILC area. The high-pass characteristic of the D^{α} term ($(j\omega)^{\alpha}$) was applied to compromise the low-pass characteristic of the controlled system. To allow for the practical applications, the implementation of $(j\omega)^{\alpha}$ was discussed as well. Besides, given a manipulator model, it was verified numerically that the optimal ILC scheme was pointed to a fractional order one[25]. Since then, many fractional-order ILC problems have been presented aiming at enhancing the performance of ILC scheme for linear or nonlinear systems[26–29]. Recently, a number of new questions emerged from FOILC such as the time domain analysis, the applications to fractional-order linear and nonlinear systems, the tuning and auto-tuning rules, and the combination with various robust feedback control strategies,

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etc[2]. The time domain analysis of FOILC addresses FO systems and the FO updating laws. In earlier studies, it was proved that the convergence conditions of the FOILC were exactly the same with the ILC cases if the FO system and the FO updating law shared the same order α[27]. The good news was that the FOILC could be directly applied to the more complicated FO systems using the same convergence conditions with the corresponding FOILC schemes. Many nice results were derived under the known knowledge of α. However, some reviewers pointed out that the above strategy might be unpractical, because the accurate identification of the system order α might be extremely hard in reality. Therefore, the order adaptive problems were raised in [12, 25]. The rejection of the order uncertainty or the order disturbance in both time and frequency domains became a specific topic in FOILC. But, as far as the authors know, the knowledge of adaptive control cannot be directly applied to these problems, and therefore new control strategies should be investigated. Several papers and their references are cited here to illustrate other recent discussions of FOILC[27,30–34]. It has been shown that the FOILC not only retains the advantages of the classical ILC, but also offers potentials for better performances in a variety of complex physical processes.

In this note, the time domain and frequency domain analyses of the linear ILC schemes are extended to a corresponding FOILC scheme. The convergence conditions are derived in time domain. Based on the result, a new FOILC scheme is proposed, which fills a gap in the previous scheme. The equivalence, adaptiveness, robustness and band-stop proper-ties of the discussed FOILC scheme are analyzed in time domain. Several numerical examples are illustrated to validate the concepts.

2 The convergence analysis in frequency domain

To date, most ILC schemes are linear ones. Thus it is common to analyze them in frequency domain, even through it is a finite time horizon problem[19,26]. Given a control system $G(s)$, the input-output relationship at the k iteration can be described by

$$Y_k(s) = G(s)U_k(s), \tag{1}$$

where $y_k(0) = y_{\rm d}(0)$, $y_{\rm d}(t)$ is the desired output. Moreover, let the learning update law in the Laplace domain be

$$U_{k+1}(s) = U_k(s) + \gamma H(s)E_k(s), \tag{2}$$

where γ is a scalar learning gain, and $H(s)$ is the learning compensator in the Laplace domain. It follows from Eqs.(1) and (2) that $E_{k+1}(s) = [1 - \gamma G(s)H(s)]\, E_k(s)$, where the commutative property of γ and $G(s)$ should be satisfied in the Laplace domain. It can be proved that the convergence condition of the above ILC scheme is

$$|1 - \gamma G(j\omega)H(j\omega)| < 1. \tag{3}$$

It can be easily verified that most ILC and FOILC learning update laws fall into the above scheme. However, there are two problems in the above discussions. Firstly, if γ is a time varying constant or even an output of a dynamic system, (3) does not hold any more. Moreover, in reality, most systems cannot be fully returned or repositioned to the desired initial state $y_{\rm d}(0)$ so that the initial disturbance must be considered.

3 Time domain analysis of fractional-order iterative learning control

Firstly, in time domain, the system can be extended to a nonlinear one

$$y^{(\alpha)}(t) = f(t, y, u), \tag{4}$$

where $\alpha \in (0, 1)$, $y(0) \in \mathbb{R}^{n\times 1}, u \in \mathbb{R}^{m\times 1}$, $\cdot^{(\alpha)}$ denotes the α order Riemann-Liouville or Caputo derivative with respect to t, and the piecewise continuous function f satisfies

$$\|\frac{\partial f}{\partial y}\|_\infty \leqslant l_1\|y\|_\infty,\ \|\frac{\partial f}{\partial u}\|_\infty \leqslant l_2\|u\|_\infty, \tag{5}$$

where $l_1, l_2 > 0$ and $\|\text{matrix}\|_\infty$ and $\|\text{vector}\|_\infty$ denote respectively the matrix norm induced by the vector $p-$norm and the maximum norm of a vector.

Remark 1 It can be seen that any fractional-order system with rational or commensurate orders can be included in Eq.(4). Moreover, it follows from (5) and the uniqueness and existence theorem of the fractional-order differential equations that, for the fixed $y(0)$ and $u(t)$, there exists a unique solution for system (4)[35–36].

Thus in each iteration,

$$\begin{cases} y_k^{(\alpha)}(t) = f(t, y_k, u_k), \\ y_{\rm d}^{(\alpha)}(t) = f(t, y_{\rm d}, u_{\rm d}), \\ e_k(t) = y_{\rm d}(t) - y_k(t), \end{cases} \tag{6}$$

where $u_{\rm d}(t)$ and $y_{\rm d}(t)$ denote the desired control effort and system output, respectively. The main objective of ILC is to find a suitable learning law so that $\lim\limits_{k\to\infty} u_k(t) = u_{\rm d}(t)$. Then it follows from the uniqueness and existence of Eq.(4) that $\lim\limits_{k\to\infty} y_k(t) = y_{\rm d}(t)$ holds almost everywhere on $t \in [0, T]$. Moreover, it follows from Eq.(6) that

$$\begin{aligned} &f_{\rm d} - f_k = f(t, y_{\rm d}, u_{\rm d}) - f(t, y_k, u_k) = \\ &f(t, y_{\rm d}, u_k) - f(t, y_k, u_k) + f(t, y_{\rm d}, u_{\rm d}) - f(t, y_{\rm d}, u_k) = \\ &\{\frac{\partial f_i}{\partial y_j}\}_{y_j=\eta_{ij}(t)} e_k(t) + \{\frac{\partial f_i}{\partial u_l}\}_{u_l=\xi_{il}(t)} \delta u_k(t) = \\ &{}_kA(t)e_k(t) + {}_kB(t)\delta u_k(t), \end{aligned} \tag{7}$$

where ${}_kA(t) \in \mathbb{R}^{n\times n}$, ${}_kB(t) \in \mathbb{R}^{n\times m}$ are depending on the iteration k due to the heredity of fractional-order systems, and there exist functions $\xi_{ij}(t)$ and $\eta_{il}(t)$ satisfying the mean value theorem.

Lemma 1 For the fractional-order nonlinear systems (6) and an arbitrary positive constant $q > 1/\alpha$, suppose $\|\frac{\partial f}{\partial u}\|_\infty \leqslant \gamma\|u\|_\infty$, we have that there exists a large enough λ satisfying $\|e_k\|_\lambda \leqslant O(\lambda^{-1/q})\|\delta u_k\|_\lambda$, where

$$O(\lambda^{-1/q}) = \frac{\gamma(1 - {\rm e}^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\varGamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}{(q\lambda)^{1/q}\varGamma(\alpha) - c(1 - {\rm e}^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\varGamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}.$$

Proof The proof can be found in [30].

Lemma 2 For the linear operator $\mathfrak{K}\{w(t)\} = \int_0^t h(t-\tau)w(\tau)\mathrm{d}\tau$, it is a bounded linear operator if and only if there exists a positive constant c satisfying

$$\int_0^t |h(\tau)|\mathrm{d}\tau \leqslant c,\ t\in[0,T], \tag{8}$$

where $|*|$ denotes the absolute value of $*$. Moreover, for an arbitrary vector $w(t)\in\mathbb{R}^n$, we have $\|\mathfrak{K}\{w\}\|_\infty \leqslant c\|w\|_\infty$.

Proof The proof can be found in [36–37].

The fractional-order ILC updating law to be discussed in this part is

$$\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{\Gamma}(t)e_k(t)\} + \Gamma(t)h(t)\cdot e_k^{(\alpha)}(t), \tag{9}$$

where $\alpha\in(0,1)$ is the system order [1], $\mathfrak{B}$ is a bounded linear operator, and $\Gamma(t)$ can be a scalar variable or a time varying gain matrix with proper dimensions.

Theorem 1 For the fractional-order system (4) and the FOILC scheme (6) and (9), if ${}_kB(t),\ k=0,1,2,\cdots$ are full column rank matrices and there exists a constant $\rho_1\in[0,1)$ satisfying

$$\|I - {}_{k+1}B(t)\Gamma(t)\|_\infty \leqslant \frac{1}{\int_0^T h(\tau)\mathrm{d}\tau}[\rho_1 - \|\Theta(t)\|_\infty\cdot\int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau - \|\Theta(t)-I\|_\infty\int_0^T|h(\tau)|\mathrm{d}\tau], \tag{10}$$

where $k\in\{0,1,2,\cdots\}$, we have $\lim\limits_{k\to\infty} y_k(t) = y_\mathrm{d}(t)$.

Proof If ${}_kB(t),\ k=0,1,2,\cdots$, are full column rank matrices, it can be proved that

$$\begin{aligned}&{}_{k+1}B(t)\delta u_{k+1}(t) = \\ &\Theta(t){}_kB(t)\delta u_k(t) - {}_{k+1}B(t)\Gamma(t)h(t)\cdot[{}_kB(t)\delta u_k(t)] - \\ &{}_{k+1}B(t)\mathfrak{B}\{\tilde{\Gamma}(t)e(t)\} - {}_{k+1}B(t)\Gamma(t)h(t)\cdot[{}_kA(t)e_k(t)],\end{aligned}$$

where $\Theta_{m\times m}(t) = {}_{k+1}B(t)[{}_kB^\mathrm{T}(t)\,{}_kB(t)]^{-1}{}_kB^T(t)$. It can be proved that

$$\begin{aligned}&{}_{k+1}B(t)\delta u_{k+1}(t) = \\ &\Theta(t)\{{}_kB(t)\delta u_k(t) - h(t)\cdot[{}_kB(t)\delta u_k(t)]\} + \\ &[\Theta(t)-I]h(t)\cdot[{}_kB(t)\delta u_k(t)] + [I - {}_{k+1}B(t)\Gamma(t)]\cdot \\ &h(t)\cdot[{}_kB(t)\delta u_k(t)] - {}_{k+1}B(t)\mathfrak{B}\{\tilde{\Gamma}(t)e(t)\} - \\ &{}_{k+1}B(t)\Gamma(t)h(t)\cdot[{}_kA(t)e_k(t)].\end{aligned}$$

Applying the $\lambda-$norm to the above equation, it follows from Lemma 1 and ${}_kB(t)(k=0,1,2,\cdots)$ are full column rank matrices that

$$\|{}_{k+1}B(t)\delta u_{k+1}(t)\|_\lambda \leqslant [\rho_1 + O(\lambda^{-1/q})]\|{}_kB(t)\delta u_k(t)\|_\lambda,$$

where

$$\begin{aligned}\rho_1 =\ &\|\Theta(t)\|_\infty\int_0^t|\tilde{h}(\tau)|\mathrm{d}\tau + \|\Theta(t)-I\|_\infty\int_0^t|h(\tau)|\mathrm{d}\tau + \\ &\|I - {}_{k+1}B(t)\Gamma(t)\|_\infty\int_0^t h(\tau)\mathrm{d}\tau.\end{aligned}$$

Therefore, if $\rho_1 < 1$, there must exist a sufficient large λ satisfying $\lim\limits_{k\to\infty}\|{}_kB(t)\delta u_k(t)\|_\lambda = 0$. It then follows from ${}_kB(t)(k=0,1,2,\cdots)$ are full column rank matrices and the uniqueness and existence of the system equation that

$$\lim_{k\to\infty} y_k(t) = y_\mathrm{d}(t),\ t\in[0,T].$$

Remark 2 In Theorem 1, the term $\mathcal{L}\{h(s)\}s^\alpha$ is corresponding to the $H(s)$ in Section 2. Moreover, it should be noted that, if $h(t)=\delta(t)$ is the Dirac-Delta function or $K(t)=K$ is a constant matrix, Eqs.(11) and (9) are the same. Moreover, if $h(t) = \dfrac{t^q}{\Gamma(q+1)}$, $q>-1$, Eq.(9) becomes

$$\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{K}(t)e(t)\} + K(t)e_k^{(\beta)}(t),$$

where $\beta = \alpha - q - 1 < \alpha$, which is corresponding to the PD$^\beta$ ILC scheme[27]. Lastly, the linear term $\mathfrak{B}\{\tilde{\Gamma}(t)e_k(t)\}$ does not influence the ILC convergence, but it is closely related to the convergence speed of the scheme.

4 A novel fractional order iterative learning control

It can be seen from Theorem 1 that the gain γ has been extended to a time varying one. Nevertheless, the convergence condition is still too complicated to be practical, because both sides of the inequality are related to the unknown terms ${}_kB(t)$. Therefore, a new FOILC scheme is proposed in this section.

Let the fractional-order ILC updating law be $\delta u_k(t) = u_\mathrm{d}(t) - u_k(t)$ and

$$\begin{aligned}&\Delta u_{k+1}(t) = u_{k+1}(t) - u_k(t) = \\ &\mathfrak{B}\{\tilde{\Gamma}(t)e(t)\} + h(t)\cdot[\Gamma(t)e_k^{(\alpha)}(t)],\end{aligned} \tag{11}$$

where $\mathfrak{B}$ is a bounded linear operator. It follows that

$$\begin{aligned}&\delta u_{k+1}(t) = \\ &\delta u_k(t) - \mathfrak{B}\{\tilde{\Gamma}(t)e(t)\} - h(t)\cdot[\Gamma(t)e_k^{(\alpha)}(t)] = \\ &\mathcal{L}^{-1}\{1-h(s)\}\cdot\delta u_k(t) + h(t)\cdot\{[I-\Gamma(t)\,{}_kB(t)]\cdot \\ &\delta u_k(t)\} - \mathfrak{B}\{\tilde{\Gamma}(t)e(t)\} - h(t)\cdot[\Gamma(t)\,{}_kA(t)e_k(t)],\end{aligned}$$

where $h(s)$ is the Laplace transform of $h(t)$ [2] and $\mathcal{L}^{-1}\{1-h(s)\}$ denotes the inverse Laplace transform of $1-h(s)$.

Applying the maximum norm to both sides of the above equation yields

$$\begin{aligned}&\|\delta u_{k+1}(t)\|_\infty \leqslant \\ &\|\mathcal{L}^{-1}\{1-h(s)\}\cdot\delta u_k(t)\|_\infty + \\ &\|h(t)\cdot\{[I-\Gamma(t)\,{}_kB(t)]\delta u_k(t)\}\|_\infty + \\ &\|\mathfrak{B}\{\tilde{\Gamma}(t)e(t)\}\|_\infty + \|h(t)\cdot[\Gamma(t)\,{}_kA(t)e_k(t)]\|_\infty.\end{aligned}$$

It follows from Lemma 2 that

$$\begin{aligned}&\|\delta u_{k+1}(t)\|_\infty \leqslant \\ &\int_0^t|\tilde{h}(\tau)|\mathrm{d}\tau\cdot\|\delta u_k(t)\|_\infty + b\|\tilde{\Gamma}(t)\|_\infty\cdot\|e(t)\|_\infty + \\ &\int_0^t|h(\tau)|\mathrm{d}\tau\cdot\|I-\Gamma(t)\,{}_kB(t)\|_\infty\cdot\|\delta u_k(t)\|_\infty + \\ &\int_0^t|h(\tau)|\mathrm{d}\tau\cdot\|\Gamma(t)\,{}_kA(t)\|_\infty\cdot\|e_k(t)\|_\infty.\end{aligned}$$

[1] For clarity, we first assume the system order α is known. The unknown order case will be discussed later.

[2] Note here, instead of $H(s)$, the $h(s)$ is denoted as the Laplace transform of $h(t)$ to avoid the confusion of $H(s)$ in Section 2 and to emphasis that $h(t)$ is a scalar function.

where $\tilde{h}(t) = \mathcal{L}^{-1}\{1 - h(s)\}$. Then applying the λ-norm to the above equation yields

$$\|\delta u_{k+1}(t)\|_\lambda \leqslant \int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau \cdot \|\delta u_k(t)\|_\lambda +$$
$$\int_0^T |h(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|I - \Gamma(t)\,{}_kB(t)\|_\infty \cdot \|\delta u_k(t)\|_\lambda +$$
$$b \max_{t\in[0,T]} \|\tilde{\Gamma}(t)\|_\infty \cdot \|e(t)\|_\lambda +$$
$$\int_0^T |h(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|\Gamma(t)\,{}_kA(t)\|_\lambda \cdot \|e_k(t)\|_\lambda.$$

By using the Lemma 1, we arrive at

$$\|\delta u_{k+1}(t)\|_\lambda \leqslant [\rho + O(\lambda^{-1/q})]\|\delta u_k(t)\|_\lambda,$$

where

$$\rho = \int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau + \int_0^T |h(\tau)|\mathrm{d}\tau \max_{t\in[0,T]} \|I - \Gamma(t)\,{}_kB(t)\|_\infty.$$

It can be seen from the above discussions that if $0 \leqslant \rho < 1$, there must exist a large enough λ satisfying $\rho + O(\lambda^{-1/q}) < 1$, which implies that $\lim_{k\to\infty} \|\delta u_k(t)\|_\lambda = 0$ for all $t \in [0,T]$. In order to derive the convergence condition of the generalized fractional-order iterative learning control scheme, the following theorem is introduced.

Theorem 2 For the fractional-order system (4) and the generalized fractional-order iterative learning control scheme (6) and (11), if there exists a constant $\rho \in [0,1)$ satisfying

$$\|I - \Gamma(t)\,{}_kB(t)\|_\infty \leqslant \frac{\rho - \int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau}{\int_0^T |h(\tau)|\mathrm{d}\tau}, \tag{12}$$

where $k \in \{0,1,2,\cdots\}$, we have $\lim_{k\to\infty} y_k(t) = y_\mathrm{d}(t)$ on $[0,T]$.

Proof It can be easily seen from (12) that

$$\int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau + \int_0^T |h(\tau)|\mathrm{d}\tau \cdot \|I - \Gamma(t)\,{}_kB(t)\|_\infty \leqslant \rho$$

holds for all $t \in [0,T]$ and $k \in \{0,1,2,\cdots\}$. In other words, $\int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau + \int_0^T |h(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|I - \Gamma(t)\,{}_kB(t)\|_\infty \leqslant \rho < 1$. Therefore,

$$\|\delta u_{k+1}(t)\|_\lambda \leqslant [\rho + O(\lambda^{-1/q})]\|\delta u_k(t)\|_\lambda,$$

and there exists a sufficient large λ satisfying $[\rho + O(\lambda^{-1/q})] < 1$, which implies that, for all $t \in [0,T]$, $\lim_{k\to\infty} \|\delta u_k(t)\| = 0$. Moreover, it follows from $y_\mathrm{d}(0) = y_k(0)$ and the uniqueness and existence of the system equation that $\lim_{k\to\infty} y_k(t) = y_\mathrm{d}(t),\ t \in [0,T]$.

Remark 3 In Eq.(11), if an extra term is added, i.e.

$$\Delta u_{k+1}(t) = h_1(t) \cdot [\Gamma_1(t)e_k^{(\alpha)}(t)] + h_2(t) \cdot [\Gamma_2(t)e_k^{(\alpha)}(t)] + \mathfrak{B}\{\tilde{\Gamma}(t)e(t)\},$$

it follows that $\|\delta u_{k+1}(t)\|_\lambda \leqslant [\hat{\rho} + O(\lambda^{-1/q})]\|\delta u_k(t)\|_\lambda$, where

$$\hat{\rho} = \int_0^T |\hat{h}(\tau)|\mathrm{d}\tau +$$
$$\int_0^T |h_1(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|I - \Gamma_1(t)\,{}_kB(t)\|_\infty +$$
$$\int_0^T |h_2(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|I - \Gamma_2(t)\,{}_kB(t)\|_\infty,$$

and $\hat{h}(t) = \mathcal{L}^{-1}\{1 - h_1(s) - h_2(s)\}$. Therefore, the convergence condition becomes

$$\max_{i=1,2}\{\|I - \Gamma_i(t)\,{}_kB(t)\|_\infty\} \leqslant \frac{\hat{\rho} - \int_0^T |\hat{h}(\tau)|\mathrm{d}\tau}{\int_0^T |h_1(\tau)|\mathrm{d}\tau + \int_0^T |h_2(\tau)|\mathrm{d}\tau}, \tag{13}$$

where $\hat{\rho} \in [0,1)$ and $k \in \{0,1,2,\cdots\}$.

In addition, if $\Delta u_{k+1}(t) = \sum_{i=1}^{N} h_i(t) \cdot [\Gamma_i(t)e_k^{(\alpha)}(t)] + \mathfrak{B}\{\tilde{\Gamma}(t)e(t)\}$, the convergence condition is that there exists a constant $\check{\rho} \in [0,1)$ satisfying

$$\max_i\{\|I - \Gamma_i(t)\,{}_kB(t)\|_\infty\} \leqslant \frac{\check{\rho} - \int_0^T |\check{h}(\tau)|\mathrm{d}\tau}{\sum_i \int_0^T |h_i(\tau)|\mathrm{d}\tau}, \tag{14}$$

where $t \in [0,T]$, $k \in \{0,1,2,\cdots\}$ and $\check{h}(t) = \mathcal{L}^{-1}\{1 - \sum_i h_i(s)\}$.

It can be seen that the convergence conditions in Theorem 2 and Remark 3 are much simpler than the ones in Theorem 1.

5 The case of constant learning gain

If $\Gamma(t) = \Gamma$ is time invariant, it follows from $\Gamma h(t) \cdot e^{(\alpha)}(t) = h(t) \cdot \left[\Gamma e^{(\alpha)}(t)\right]$ that either Eq.(10) or (12) guarantees the convergence of the fractional-order learning schemes, where

$$\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{\Gamma}(t)e_k(t)\} + \Gamma h(t) \cdot e_k^{(\alpha)}(t).$$

Proof The proof comes from Theorems 1 and 2.

Remark 4 Let $h(t) = \delta(t)$ be the Dirac-Delta function, it can be seen from Theorems 1 and 2 that the convergence condition is either $\|I_m - \Gamma(t)\,{}_kB(t)\|_\infty < 1$, where $k \in \{0,1,2,\cdots\}$ or $\|I_n - {}_{k+1}B(t)\Gamma(t)\|_\infty < 1$, where $k \in \{0,1,2,\cdots\}$, ${}_kB(t)$ and ${}_{k+1}B(t)$ are full column rank matrices, and the ILC updating law is

$$\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{\Gamma}(t)e_k(t)\} + \Gamma(t)e_k^{(\alpha)}(t). \tag{15}$$

In other words, Eq.(15) guarantees the convergence of the ILC scheme if one of the following conditions is satisfied:

- There exists a $\Gamma(t)$ satisfying $\|I_m - \Gamma(t)\,{}_kB(t)\|_\infty < 1$, where $k \in \{0,1,2,\cdots\}$.
- There exists a $\Gamma(t)$ satisfying $\|I_n - {}_{k+1}B(t)\Gamma(t)\|_\infty < 1$, where $k \in \{0,1,2,\cdots\}$, $\mathrm{rank}({}_{k+1}B(t)) = m$ and ${}_{k+1}B(t) \in \mathbb{R}^{n\times m}$.

Based on the discussions in Sections 3 and 4, some further results are presented in the following.

5.1 An adaptive fractional order iterative learning control

Given a linear fractional-order system, in frequency domain, it follows from Eq.(1) that the system order α is absorbed into $G(s)$ so that the learning filter $H(s)$ can be investigated directly. For example, besides the frequency domain methods, the time domain analysis previously discussed in this paper can be applied as well. However, in Theorems 1 and 2, the system order α exists in the updating law, which means that the FOILC is still dependant on the accurate value of α, which maybe impractical. Therefore, to allow for the real application and the consistence of time and frequency analysis, an adaptive FOILC scheme is proposed based on the results in Section 4.

In Theorem 2, if $\Gamma(t)=\Gamma$ is a constant learning gain, the fractional-order learning law and the convergence condition become

$$\Delta u_{k+1}(t)=\mathfrak{B}\{\tilde{\Gamma}(t)e_k(t)\}+\Gamma h(t)\cdot e_k^{(\alpha)}(t), \tag{16}$$

and

$$\|I-\Gamma_k B(t)\|_\infty \leqslant \frac{\rho-\int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau}{\int_0^T|h(\tau)|\mathrm{d}\tau}. \tag{17}$$

Moreover, if α is unknown and $0<a\leqslant\alpha\leqslant b\leqslant 1$, let $\Phi(s)=H(s)s^\beta$, $\Delta u_{k+1}(t)=\mathfrak{B}\{\tilde{\Gamma}(t)e_k(t)\}+\Gamma\mathcal{L}^{-1}\{\Phi(s)\}\cdot e_k(t)$, where $H(s)=\mathcal{L}\{h(t)\}$ and the given constant $\beta\in[a,b]$. It follows from (17) that the convergence condition becomes

$$\|I-\Gamma_k B(t)\|_\infty \leqslant \frac{\rho-\int_0^T|\tilde{\phi}(\tau)|\mathrm{d}\tau}{\int_0^T|\phi(\tau)|\mathrm{d}\tau}, \tag{18}$$

where $\phi(t)=\mathcal{L}^{-1}\{\Phi(s)s^{-\xi}\}$, $\tilde{\phi}(t)=\mathcal{L}^{-1}\{1-\Phi(s)s^{-\xi}\}$ and $\xi\in\{a,b\}$.

An example is presented for illustration here. For the fractional-order nonlinear system (4), where $0<a\leqslant\alpha\leqslant b\leqslant 1$, and the FOILC scheme (6) and

$$u_{k+1}(t)=u_k(t)+\Gamma\hat{\phi}(t)\cdot e(t).$$

let $\hat{\phi}(t)=\lambda t^{\mu-1}E_{\beta,\mu}(-\lambda t^\beta)$, where $\lambda>0$, $\beta\in(0,1]$ and $\mu\in(-a,1-b]$, it follows from Theorem 2 that the convergence condition becomes

$$\|I-{}_kB(t)K(t)\|_\infty \leqslant \frac{\rho-\int_0^T|\tilde{\phi}(\tau)|\mathrm{d}\tau}{\int_0^T|\phi(\tau)|\mathrm{d}\tau},$$

where $\rho\in(0,1)$, $\phi(t)=\lambda t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)$, and $\tilde{\phi}(\tau)=\dfrac{\mathrm{d}E_\beta(-\lambda t^\beta)}{\mathrm{d}t}<0$. Moreover, for the arbitrary positive constants T and $\rho\in(0,1)$, there exists a small enough λ satisfying the convergence condition.

Remark 5 In the above discussion, $\mu\in(-a,1-b]$ is called the 'learnable domain' of α. Particularly, for an unknown $\alpha\in(0,1]$, we can always let $\mu=0$, i.e. $\hat{\phi}(t)=\lambda\dfrac{\mathrm{d}E_\beta(-\lambda t^\beta)}{\mathrm{d}t}$, so that the convergence condition can always be guaranteed, i.e. the learnable domain covers all the possibilities of α.

5.2 The desirable band-stop FOILC scheme

Let the FOILC learning law be

$$\Delta u_{k+1}=\Gamma\phi(t)\cdot e^{(\alpha)}(t), \tag{19}$$

where $\phi(t)=\dfrac{a}{\lambda}t^{\beta-1}E_{\beta,\beta}(-\dfrac{at^\beta}{\lambda})+\dfrac{\mathrm{d}}{\mathrm{d}t}E_{\tilde{\beta}}(-\dfrac{ct^{\tilde{\beta}}}{b})+\delta(t)$, $\delta(t)$ denotes the Dirac-Delta function, and $\mathcal{L}\{\phi(t)\}=\dfrac{b}{b+\lambda s^\beta}+\dfrac{as^{\tilde{\beta}}}{c+as^{\tilde{\beta}}}$. It follows from Remark 3 that the convergence condition of the above learning law is

$$\|I-\Gamma_k B(t)\|_\infty < \frac{\tilde{\rho}-\int_0^T|\frac{\mathrm{d}}{\mathrm{d}t}[E_\beta(-\frac{bt^\beta}{\lambda})-E_{\tilde{\beta}}(-\frac{ct^{\tilde{\beta}}}{a})]|\mathrm{d}t}{\frac{a}{\lambda}T^\beta E_{\beta,\beta+1}(-\frac{aT^\beta}{\lambda})-E_{\tilde{\beta}}(-\frac{cT^{\tilde{\beta}}}{b})+1}, \tag{20}$$

where $\tilde{\rho}\in[0,1)$.

Remark 6 It can be seen from the above discussion that $\Phi(s)=\mathcal{L}\{\phi(t)\}$ is a summation of a low-pass filter and a high-pass filter. Besides, $|\Phi(s)|=1$ for both $s\to 0$ and $s\to\infty$. Therefore, compared with the desirable unit-gain compensator[26], $\Phi(s)$ is called the "desirable band-stop" FOILC compensator. The idea of the "desirable band-stop" method makes some practical sense. For example, in reality, the noise and some other disturbances and uncertainties in the feedback loop can lead to the divergence of the ILC scheme, even if the convergence conditions are satisfied[2]. Therefore a band-stop compensator can efficiently reduce the influences of the certain noises. Besides, refer to the hereditary property of the fractional order systems, the initial value $y_k(0)$ can not be fully reset to $y_\mathrm{d}(0)$, but may vary around it in a certain bound. Thus the initial disturbances must be considered. It follows from the finite energy property of the band-stop compensator that the tracking error is bounded so as the initial disturbances.

Remark 7 Given the linear fractional order systems $y^{(\beta)}(t)=-\dfrac{b}{\lambda}y(t)+\dfrac{b}{\lambda}x(t)$, and $y^{(\tilde{\beta})}(t)=-\dfrac{c}{a}y(t)+x^{(\tilde{\beta})}(t)$, it can be easily proved that the transfer functions of the above two equations are $Y(s)=\dfrac{b}{b+\lambda s^\beta}X(s)$, and $Y(s)=\dfrac{as^{\tilde{\beta}}}{c+as^{\tilde{\beta}}}X(s)$, respectively. Both of them can be easily realized using the idea of fractional order element networks[38].

6 Simulation illustrations

In this section, a fractional-order nonlinear system associated with the desirable band-stop FOILC scheme are numericallly illustrated in MATLAB/Simulink. A band-stop white noise is added to the system and a number of plots are provided to illustrate our main results.

The fractional order nonlinear system is

$$y_k^{(1/2)}(t)=y_k^{(8/9)}(t)+u_k(t),\ k=0,1,2,\cdots, \tag{21}$$

where the reference $y_\mathrm{d}(t)=12t(1-t)$ and $y_k(0)=0$.

Moreover, in Laplace domain, the FOILC updating law is shown as

$$U_{k+1}(s)=U_k(s)+\frac{9}{10}\Phi(s)s^{(\xi)}E(s), \tag{22}$$

where $\xi\in\{\dfrac{2}{5},\dfrac{1}{2}\}$ and $\Phi(s)=\dfrac{1}{1+s^{1/2}}+\dfrac{s^{1/5}}{1+s^{1/5}}$.

It can be seen from Fig.1 and 2 that the FOILC updating law (22) works for both $\xi=\dfrac{1}{2}$ and $\xi=\dfrac{2}{5}$. Nevertheless, it is shown that the convergence speed is faster for $\xi=\dfrac{1}{2}$, which is the same with the system order. Moreover, if $\xi>\dfrac{1}{2}$ the monotonicity of the tracking errors or the convergence of the scheme may not be satisfied, which is also discussed in Remark 2. Lastly, the linear term

$\mathfrak{B}\{\tilde{\Gamma}(t)e(t)\}$ does not influence the convergence condition. However, it is strongly related to the convergence speed of the FOILC scheme, which implies that a proper linear operator $\mathfrak{B}$ should be used to guarantee the monotone decreasing of the tracking errors. The implementations of fractional order operators can be found in [39,40].

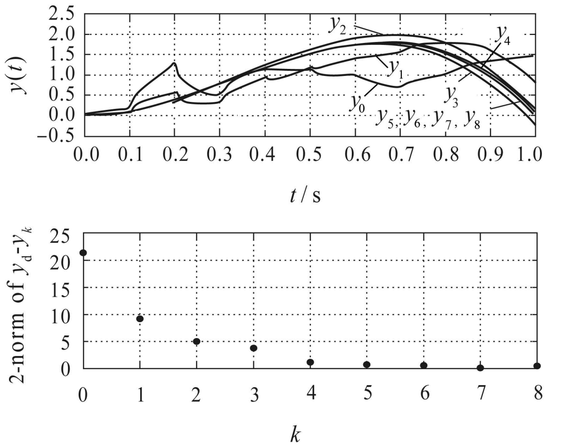

Fig. 1 The system outputs $y_k(t)$ and the $2-$norms of $e_k(t)$ for different iterations, where $k \in \{0, 1, 2, \cdots, 8\}$ and $\xi = \frac{1}{2}$.

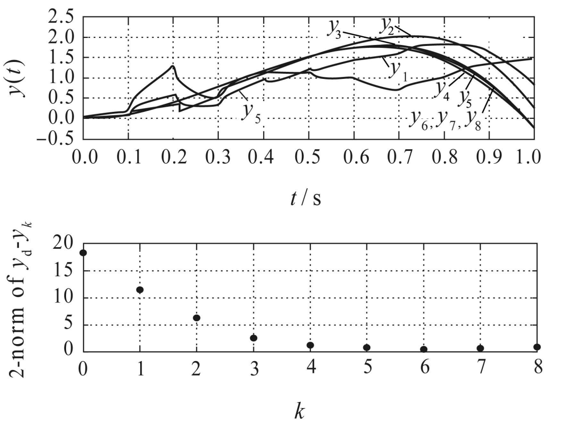

Fig. 2 The system outputs $y_k(t)$ and the $2-$norms of $e_k(t)$ for different iterations, where $k \in \{0, 1, 2, \cdots, 8\}$ and $\xi = \frac{2}{5}$.

7 Conclusions

In this paper, the frequency domain analysis of the ILC scheme and its corresponding time domain analysis of an FOILC scheme were discussed. Based on our time domain results, a novel FOILC scheme is proposed, which greatly simplifies the ILC convergence conditions. The constant learning gain cases are investigated which provides a number of new insights of FOILC. The equivalence of the convergence conditions is proved so that the convergence of the FOILC scheme can be easily guaranteed depending on the controller information. The order adaption problem, a distinguishing feature for fractional order systems, is well solved in time domain. Refer to the practical applications of FOILC, the desirable band-stop FOILC scheme is proposed and a number of numerical simulation illustrations are presented.

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