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$\mathcal{L}_1$ adaptive control with sliding-mode based adaptive law

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Abstract

This paper presents an adaptive control scheme with an integration of sliding mode control into the $\mathcal{L}_1$ adaptive control architecture, which provides good tracking performance as well as robustness against matched uncertainties. Sliding mode control is used as an adaptive law in the $\mathcal{L}_1$ adaptive control architecture, which is considered as a virtual control of error dynamics between estimated states and real states. Low-pass filtering mechanism in the control law design prevents a discontinuous signal in the adaptive law from appearing in actual control signal while maintaining control accuracy. By using sliding mode control as a virtual control of error dynamics and introducing the low-pass filtered control signal, the chattering effect is eliminated. The performance bounds between the close-loop adaptive system and the closed-loop reference system are characterized in this paper. Numerical simulation is provided to demonstrate the performance of the presented adaptive control scheme.

Keywords: $\mathcal{L}_1$ adaptive control, sliding-mode control, matched uncertainties

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1 Introduction

Sliding mode control has been successfully implemented for nonlinear systems with matched uncertainties. In sliding mode control, trajectories of the system are forced to and maintained on the sliding surface. On the sliding surface, the system has a desired behavior, which is independent of matched uncertainties. Despite this robustness, the implementation of sliding mode control suffers from a chattering phenomenon characterized by oscillation of system trajectories around the sliding surface, which is due to the imperfections in switching devices and delays [1]. In order to solve this chattering problem, a number of approaches have been proposed to improve sliding mode control. One approach is to replace the discontinuous control law with a continuous one by which the system motion is restricted within a boundary layer of the sliding surface [2, 3]. This boundary layer solution is a trade-off between control performance and chattering attenuation. Another approach to avoid chattering is higher-order sliding mode control in which the discontinuous control signal appears on the higher-order derivatives of the sliding variable instead of the first-order deriva-

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tive [4–6]. Higher-order sliding mode approach requires higher-order derivatives of the sliding variable, which could be its limitation [7]. Adding an integrator for sliding mode control signal is another approach to solve the chattering problem [8]. In this approach, the sliding variable contains unknown disturbance as presented in [8], and therefore requires a disturbance estimator, which increases the complexity of sliding mode control design.

This paper presents an adaptive control scheme by introducing sliding mode control to the adaptive law design in the $\mathcal{L}_1$ adaptive control architecture, which provides good tracking performance as well as robustness without introducing the chattering effect from the sliding mode control. The $\mathcal{L}_1$ adaptive control is a variation of the model reference adaptive control with improved transient performance and handling of time-varying parameters and uncertainties. The architecture of the $\mathcal{L}_1$ adaptive control consists of a state predictor, an adaptive law, and a control law. The adaptive law is used to update the parameters in the state predictor to make the estimated states arbitrarily close to the real states. The typical adaptive law design for the $\mathcal{L}_1$ adaptive control includes projection-type adaptive law [9] and piece-wise constant adaptive law [10,11]. In this paper, an alternative adaptive law design using sliding mode control is provided, which makes the state predictor provide a good estimation of real states independent of matched uncertainties. Chattering effect will not appear in the state predictor dynamics, since sliding-mode based adaptive law is used as a virtual control of error dynamics without any physical actuator involved. The initial states of error dynamics can be chosen on the sliding surface, which ensures robustness from the initial time and further improve the performance. The low-pass filtering design in the control law prevents a discontinuous signal in the adaptive law from appearing in the actual control signal while maintaining the control accuracy. By using sliding mode control as a virtual control of error dynamics and introducing the low-pass filtered control signal, the chattering effect is eliminated in the presented control scheme.

The contribution of this paper lies in two parts. The first part is that the $\mathcal{L}_1$ adaptive control architecture can be borrowed by the sliding mode control for avoiding chattering effect. Instead of applying sliding mode control to the real plant, it is applied to the state predictor (virtual plant) to avoid the chattering effect while maintaining robustness against matched uncertainties. The second part is that the $\mathcal{L}_1$ adaptive control architecture can have an alternative adaptive law design by using the sliding mode control. It shows the potential to use other aggressive control designs to fulfill the fast adaptation of the $\mathcal{L}_1$ adaptive control architecture. Overall, the presented adaptive control scheme with sliding-mode based adaptive law demonstrates the merging of two control algorithms and provides a better understanding of the fast and robust adaptive mechanism.

The paper is organized as follows. Section 2 gives the problem formulation. The $\mathcal{L}_1$ adaptive control architecture with sliding-mode based adaptive law is presented in Section 3. Stability and uniform performance bounds are given in Section 4, while results are verified in Section 5 through numerical simulation. Finally, Section 6 concludes this paper.

2 Problem formulation

Consider the following uncertain dynamic system:

$$\begin{cases} \dot{x}(t) = Ax(t) + b(u(t) + \sigma(x, t)), \\ y(t) = cx(t), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the output,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}$$

with known positive constants $a_1, a_2, \ldots, a_n$, $b = [0\ 0 \cdots 0\ 1]^{\mathrm{T}}$, $c \in \mathbb{R}^n$ makes $c(-A)^{-1}b$ nonzero, and $\sigma(x, t)$ is an unknown function which represents the matched uncertainty.

Remark 1 For a general controllable linear time-invariant system, there always exists a transformation in an appropriate coordinate system to make the system represented in the controllable canonical form in equation (1). The term $\sigma(x, t)$ captures not only matched uncertainties but also the mismatch part that matrix A for actual plants does not have the same constants $a_1, a_2, \ldots, a_n$. For a nonlinear system with nonlinear terms in the last row of matrix A in equation (1), it can always be transformed into a linear matrix A together with a nonlinear matched uncertainties $\sigma(x, t)$.

Assumption 1 The system is assumed to be controllable, and $\operatorname{rank}(b)$ and $\operatorname{rank}(c)$ are nonzero.

Assumption 2 The unknown function $\sigma(x,t)$ is assumed to be bounded and satisfy

$$|\sigma(x,t)| < \Phi(x,t), \tag{2}$$

where $\Phi(x,t)$ is a known function.

Assumption 3 (Semiglobal Lipschitz condition) For any $\delta > 0$, there exist $L(\delta) > 0$ and $B > 0$ such that $|\sigma(x,t) - \sigma(\bar{x},t)| \leqslant L(\delta)\|x-\bar{x}\|_\infty$, $|\sigma(0,t)| \leqslant B$, for all $\|x\|_\infty \leqslant \delta$ and $\|\bar{x}\|_\infty \leqslant \delta$ uniformly in t.

The control objective is to design a full-state feedback controller to ensure that $y(t)$ tracks the output of the desired system:

$$\begin{cases} \dot{x}_{\mathrm{des}}(t) = Ax_{\mathrm{des}}(t) + bk_{\mathrm{g}}r(t), \\ y_{\mathrm{des}}(t) = cx_{\mathrm{des}}(t), \end{cases} \tag{3}$$

where $k_{\mathrm{g}} = (c(-A)^{-1}b)^{-1}$ and $r(t)$ is the reference signal.

3 $\mathcal{L}_1$ adaptive control with sliding-mode based adaptive law

In this section, we develop an adaptive control architecture with sliding-mode based adaptive law for the system in (1). This architecture consists of three elements: state predictor, sliding-mode based adaptive law, and control law. The elements of this adaptive controller are introduced next.

State predictor

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + b(u(t) + \hat{\sigma}(t)), \ \ \hat{x}(0) = x_0, \\ \hat{y}(t) = c\hat{x}(t). \end{cases} \tag{4}$$

Let $\tilde{x} = \hat{x} - x$ and $\tilde{y} = \hat{y} - y$. Then, we obtain the following error dynamics from (1) and (4):

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + b(\hat{\sigma}(t) - \sigma(x,t)), \ \ \tilde{x}(0) = 0, \\ \dot{\tilde{y}}(t) = c\tilde{x}(t), \end{cases} \tag{5}$$

where the adaptive parameter $\hat{\sigma}$ is governed by the following adaptation law.

Adaptive law Consider $\hat{\sigma}$ as a virtual control signal in the error dynamics (5), and use sliding mode control method from [12] to design $\hat{\sigma}$ as

$$\hat{\sigma}(t) = a_1\tilde{x}_1(t) + \sum_{i=2}^{n}(a_i - m_{i-1})\tilde{x}_i(t) - \rho(x,t)\operatorname{sgn}(s(\tilde{x})), \tag{6}$$

where x_i is the ith component of states x, $\rho(x,t)$ is defined as $\rho(x,t) \geqslant \Phi(x,t) + \eta$ with designed positive scalar η, $s(\tilde{x})$ represents the sliding surface and is defined as

$$s(\tilde{x}) = \sum_{i=1}^{n-1} m_i\tilde{x}_i(t) + \tilde{x}_n(t), \tag{7}$$

and the coefficients $m_1, m_2, \dots, m_{n-1}$ constitute a Hurwitz polynomial, which guarantees a stable reduced-order sliding motion.

Control law

$$u(s) = k_{\mathrm{g}}r(s) - C(s)\hat{\sigma}(s), \tag{8}$$

where $C(s) = \dfrac{k}{s+k}$ is the low-pass filter with $k > 0$, $r(s)$ is the Laplace transformation of reference signal $r(t)$, and $\hat{\sigma}(s)$ is the Laplace transformation of $\hat{\sigma}(t)$.

For the proof of stability and uniform performance bounds, the choice of k in the low-pass filter $C(s)$ needs to ensure that there exists ρ_{r} such that

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_{\mathrm{r}} - \|H(s)k_{\mathrm{g}}\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} - \|r_0\|_{\mathcal{L}_\infty}}{L(\rho_{\mathrm{r}})\rho_{\mathrm{r}} + B}, \tag{9}$$

where $H(s) = (sI - A)^{-1}b$, $G(s) = H(s)(1 - C(s))$, $r_0 = (sI - A)^{-1}x_0$, $L(\rho_{\mathrm{r}})$ is the Lipschitz constant as defined in Assumption 3, and B is defined in Assumption 3.

Remark 2 The upper bound in (9) is a consequence of the semiglobal Lipschitz property of $\sigma(x,t)$ in Assumption 3. If $\sigma(x,t)$ is globally Lipschitz with uniform Lipschitz constant L, then $\lim\limits_{\rho_{\mathrm{r}}\to\infty} \dfrac{\rho_{\mathrm{r}} - \|H(s)k_{\mathrm{g}}\|_{\mathcal{L}_1}\cdot\|r\|_{\mathcal{L}_\infty} - \|r_0\|_{\mathcal{L}_\infty}}{L\rho_{\mathrm{r}} + B} = \dfrac{1}{L}$, and the upper bound in (9) degenerates into $\|G(s)\|_{\mathcal{L}_1} < \dfrac{1}{L}$. ρ_{r} characterizes the domain of attraction of the closed-loop reference system described in the next section. We further define ρ to characterize the domain of attraction of the system in (1), which is

$$\rho = \rho_{\mathrm{r}} + \beta, \tag{10}$$

where ρ_{r} is introduced in (9), and β is an arbitrary positive constant that satisfies $\|G(s)\|_{\mathcal{L}_1}L(\rho) < 1$. Note that (9) implies that $\|G(s)\|_{\mathcal{L}_1}L(\rho_{\mathrm{r}}) < 1$. Since $L(\rho)$ depends continuously on ρ, $\|G(s)\|_{\mathcal{L}_1}L(\rho) < 1$ can always be satisfied if β is small enough.

The $\mathcal{L}_1$ adaptive controller consists of (4), (6) and (8) subject to stability condition (9).

4 Stability analysis

This section characterizes the estimation performance from state predictor, and provides the error bounds of both states and control input between the closed-loop adaptive system and the closed-loop reference system.

Lemma 1 For the system in (1) subject to Assumption 2, and the state predictor in (4), the adaptive law in (6) can make

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leqslant \gamma_0,$$

where γ_0 can be arbitrarily small.

Proof It follows from (5) and (7) that

$$\begin{aligned}\dot{s}(\tilde{x}) &= \sum_{i=1}^{n-1} m_i\tilde{x}_{i+1} - \sum_{i=1}^{n} a_i\tilde{x}_i + \hat{\sigma}(t) - \sigma(x,t)\\ &= -a_1\tilde{x}_1(t) + \sum_{i=2}^{n}(m_{i-1} - a_i)\tilde{x}_i(t)\\ &\quad +\hat{\sigma}(t) - \sigma(x,t). \end{aligned} \tag{11}$$

Substitution of adaptive law (6) into (11) yields

$$\dot{s}(\tilde{x}) = -\rho(x,t)\mathrm{sgn}(s(\tilde{x})) - \sigma(x,t).$$

Consider the Lyapunov function candidate for the switching function $s(\tilde{x})$ as $V = \dfrac{1}{2}s^2(\tilde{x})$. Then,

$$\begin{aligned}\dot{V} &= s(\tilde{x})(-\rho(x,t)\mathrm{sgn}(s(\tilde{x})) - \sigma(x,t))\\ &\leqslant |s(\tilde{x})|(|\sigma(x,t)| - \rho(x,t))\\ &\leqslant -\eta|s(\tilde{x})|,\end{aligned} \tag{12}$$

and note that $s(0) = 0$ since $\tilde{x}(0) = 0$ in (5), therefore trajectory of system starts at the sliding surface from the initial time and will be maintained on this sliding surface independent of matched uncertainties. On the sliding surface, the motion is governed by

$$s(\tilde{x}) = \sum_{i=1}^{n-1} m_i\tilde{x}_i + \tilde{x}_n = 0.$$

Since the coefficients $m_1, m_2, \dots, m_{n-1}$ constitute a Hurwitz polynomial and $\tilde{x}(0) = 0$, $\tilde{x}(t)$ will stay on the convergence point $\tilde{x}(0) = 0$ throughout the entire response of the system, which implies that there always exists an arbitrarily small constant γ_0 such that $\|\tilde{x}\|_{\mathcal{L}_\infty} \leqslant \gamma_0$, and concludes the proof. □

Remark 3 If $\tilde{x}(0) \neq 0$ in (5), $\tilde{x}(t)$ exponentially converges to zero. From equation (12), we obtain $\sqrt{2V(t)} \leqslant \sqrt{2V(0)} - \eta t$ for $t \leqslant t_{\rm s}$ and $V(t) = 0$ for $t > t_{\rm s}$ with $t_{\rm s} = \dfrac{\sqrt{2V(0)}}{\eta}$, which implies that the trajectory of error dynamics system (5) goes to $s(\tilde{x}) = 0$ in finite-time. On the sliding surface $s(\tilde{x}) = 0$, the error dynamics becomes $\dot{\bar{x}} = A_m\bar{x}$ with $\bar{x} \in \mathbb{R}^{n-1}$, $\bar{x} = [\tilde{x}_1\ \ \tilde{x}_2\ \ \cdots\ \ \tilde{x}_{n-1}]$, $A_m \in \mathbb{R}^{(n-1)\times(n-1)}$, and $A_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -m_1 & -m_2 & -m_3 & \cdots & -m_{n-1}\end{bmatrix}$.
Then, we can further obtain $\bar{x}(t) = \mathrm{e}^{A_m t}\bar{x}(t_{\rm s})$. It means that $\bar{x}(t)$ exponentially converges to zero, which implies that $\tilde{x}$ also exponentially converges to zero. The rate of convergence can be adjusted by choice of $m_1, m_2, \dots, m_{n-1}$ in (7).

Next, we give the definition and stability analysis of the closed-loop reference system, and characterize the error bounds between the closed-loop reference system and the closed-loop adaptive system.

We define the closed-loop reference system with the nonadaptive control law as follows:

$$\begin{cases}\dot{x}_{\rm ref}(t) = Ax_{\rm ref}(t) + b(u_{\rm ref}(t) + \sigma(x_{\rm ref}, t)),\\ u_{\rm ref}(s) = k_{\rm g}r(s) - C(s)\sigma_{\rm ref}(s),\\ y_{\rm ref}(t) = cx_{\rm ref}(t),\ \ x_{\rm ref}(0) = x_0,\end{cases} \tag{13}$$

where $\sigma_{\rm ref}(s)$ is the Laplace transformation of $\sigma(x_{\rm ref}, t)$.

Lemma 2 For the closed-loop reference system in (13) subject to the stability condition (9), if $\|x_0\|_\infty < \rho_{\rm r}$, then we have

$$\|x_{\rm ref}\|_{\mathcal{L}_\infty} < \rho_{\rm r}, \tag{14}$$

$$\|u_{\rm ref}\|_{\mathcal{L}_\infty} < \rho_{\rm ur}, \tag{15}$$

where $\rho_{\rm r}$ is introduced in (9), and $\rho_{\rm ur} = |k_{\rm g}| \cdot \|r\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}(L(\rho_{\rm r})\rho_{\rm r} + B)$.

Proof We prove the bound in (14) by contradiction. If (14) is not true, since $\|x_0\|_\infty < \rho_{\rm r}$ and $x_{\rm ref}(t)$ is continuous, there exists t' such that $\|x_{\rm ref}(t')\| = \rho_{\rm r}$ while $\|x_{\rm ref}(t)\| < \rho_{\rm r},\ \forall t \in [0, t')$, which implies that

$$\|x_{{\rm ref}_{t'}}\|_{\mathcal{L}_\infty} = \rho_{\rm r}. \tag{16}$$

It follows from (13) that

$$\begin{aligned}x_{\rm ref}(s) &= G(s)\sigma_{\rm ref}(s) + H(s)k_{\rm g}r(s)\\ &\quad +(sI - A)^{-1}x_0,\end{aligned} \tag{17}$$

where $H(s) = (sI - A)^{-1}b$ and $G(s) = H(s)(1 - C(s))$. Example 5.2 in [1, Page 199] implies that

$$\begin{aligned}\|x_{{\rm ref}_{t'}}\|_{\mathcal{L}_\infty} &\leqslant \|G(s)\|_{\mathcal{L}_1}\|\sigma_{{\rm ref}_{t'}}\|_{\mathcal{L}_\infty}\\ &\quad +\|H(s)k_{\rm g}\|_{\mathcal{L}_1}\|r_{t'}\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty},\end{aligned} \tag{18}$$

where $r_0 = (sI - A)^{-1}x_0$. Using Assumption 3 and the upper bound in (16), we arrive at the following upper bound:

$$\|\sigma_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} \leqslant L(\rho_r)\rho_r + B. \quad (19)$$

The derivation of equation (19) uses the fact that $\|r_{t'}\|_{\mathcal{L}_\infty} \leqslant \|r\|_{\mathcal{L}_\infty}$. Substitution of (19) into (18) yields

$$\|x_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} \leqslant \|G(s)\|_{\mathcal{L}_1} L(\rho_r)\rho_r + \|H(s)k_g\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \times \|G(s)\|_{\mathcal{L}_1} B + \|r_0\|_{\mathcal{L}_\infty}.$$

The stability condition (9) can be solved for ρ_r to obtain the following upper bound:

$$\|G(s)\|_{\mathcal{L}_1} L(\rho_r)\rho_r + \|H(s)k_g\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} B + \|r_0\|_{\mathcal{L}_\infty} < \rho_r,$$

which implies that $\|x_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} \leqslant \rho_r$ and contradicts (16). This proves (14), which further implies that $\|\sigma_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} < L(\rho_r)\rho_r + B$ holds for all t with strict inequality. Therefore, it follows from the definition of $u_{\text{ref}}(s)$ in (13) that

$$\|u_{\text{ref}}\|_{\mathcal{L}_\infty} < |k_g| \cdot \|r\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}(L(\rho_r)\rho_r + B),$$

which proves (15), and concludes the proof. □

Theorem 1 Given the closed-loop system with the $\mathcal{L}_1$ adaptive controller defined via (4), (6), and (8), subject to the stability condition in (9), and the closed-loop reference system in (13), if $\|x_0\|_\infty < \rho_r$, then

$$\|x - x_{\text{ref}}\|_{\mathcal{L}_\infty} < \gamma_{x_e}, \quad (20)$$

$$\|u - u_{\text{ref}}\|_{\mathcal{L}_\infty} < \gamma_{u_e}, \quad (21)$$

where

$$\gamma_{x_e} = \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L(\rho)}\gamma_0 + \beta_1 < \beta, \quad (22)$$

$$\gamma_{u_e} = \|C(s)\|_{\mathcal{L}_1} L(\rho)\gamma_{x_e} + \|C(s)\frac{1}{c_0^{\mathrm{T}} H(s)} c_0^{\mathrm{T}}\|_{\mathcal{L}_1}\gamma_0. \quad (23)$$

γ_0 is introduced in Lemma 1, which can be arbitrarily small, β_1 is an arbitrarily small positive constant, and β is introduced in (10).

Proof (By contradiction) Assume that (20) and (21) are not true. Then, since $\|x(0) - x_{\text{ref}}(0)\|_\infty = 0 < \gamma_{x_e}$, $\|u(0) - u_{\text{ref}}(0)\|_\infty = 0 < \gamma_{u_e}$, and $x(t)$, $x_{\text{ref}}(t)$, $u(t)$, and $u_{\text{ref}}(t)$ are continuous, there exists t' such that $\|x(t') - x_{\text{ref}}(t')\|_\infty < \gamma_{x_e}$, or $\|u(t') - u_{\text{ref}}(t')\|_\infty < \gamma_{u_e}$, while $\|x(t) - x_{\text{ref}}(t)\|_\infty < \gamma_{x_e}$ and $\|u(t) - u_{\text{ref}}(t)\|_\infty < \gamma_{u_e}$ for all $t \in [0, t')$, which implies that

$$\|(x - x_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} = \gamma_{x_e}, \text{ or } \|(u - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} = \gamma_{u_e}. \quad (24)$$

From Lemma 2, we have

$$\|x_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} \leqslant \rho_r, \ \|u_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} \leqslant \rho_{u_r}. \quad (25)$$

It follows from (10), (22) and the bounds in (24) and (25) that

$$\|x_{t'}\|_{\mathcal{L}_\infty} \leqslant \rho_r + \gamma_{x_e} < \rho, \ \|u_{t'}\|_{\mathcal{L}_\infty} \leqslant \rho_{u_r} + \gamma_{u_e}. \quad (26)$$

The Laplace transformation of the system in (1) is

$$x(s) = G(s)\sigma(s) - H(s)C(s)\tilde{\sigma}(s) + H(s)k_g r(s) + (sI - A)^{-1}x_0, \quad (27)$$

where $\sigma(s)$ is the Laplace transformation of $\sigma(t)$, and $\tilde{\sigma}(s) = \hat{\sigma}(s) - \sigma(s)$. Let $e(t) = x(t) - x_{\text{ref}}(t)$. From (17) and (27), we obtain

$$e(s) = G(s)(\sigma(s) - \sigma_{\text{ref}}(s)) - H(s)C(s)\tilde{\sigma}(s). \quad (28)$$

It follows from (4) that

$$\tilde{x}(s) = H(s)\tilde{\sigma}(s).$$

Then, equation (28) becomes

$$e(s) = G(s)(\sigma(s) - \sigma_{\text{ref}}(s)) - C(s)\tilde{x}(s). \quad (29)$$

Assumption 3 and bounds in (25) and (26) yield

$$\|(\sigma(x, t) - \sigma(x_{\text{ref}}, t))_{t'}\|_{\mathcal{L}_\infty} \leqslant L(\rho)\|e_{t'}\|_{\mathcal{L}_\infty}. \quad (30)$$

Following from (29) and (30), we further obtain

$$\|e_{t'}\|_{\mathcal{L}_\infty} \leqslant \|G(s)\|_{\mathcal{L}_1} L(\rho)\|e_{t'}\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty}.$$

It follows from Lemma 2 and Remark 2 that

$$\|e_{t'}\|_{\mathcal{L}_\infty} \leqslant \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L(\rho)}\gamma_0 < \gamma_{x_e}.$$

From the definition in (8) and control law using in (13) one can derive

$$u(s) - u_{\text{ref}}(s) = -C(s)(\sigma(s) - \sigma_{\text{ref}}(s)) - C(s)\tilde{\sigma}(s).$$

It follows from Lemma 4 in [13] that there exists $c_0 \in \mathbb{R}^n$ such that

$$c_0^{\mathrm{T}} H(s) = \frac{N_n(s)}{N_d(s)},$$

where the order of $N_d(s)$ is one more than the order of $N_n(s)$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials. Then, we have $C(s)\tilde{\sigma}(s) = \dfrac{C(s)}{c_0^{\mathrm{T}} H(s)} c_0^{\mathrm{T}} H(s)\tilde{\sigma}(s)$. Since

$C(s)$ is stable and strictly proper, the complete system $C(s)\dfrac{1}{c_0^{\mathrm{T}}H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Hence, we have the following upper bound:

$$\begin{aligned}\|(u-u_{\mathrm{ref}})_{t'}\|_{\mathcal{L}_\infty} &\leqslant \|C(s)\|_{\mathcal{L}_1}L(\rho)\|e_{t'}\|_{\mathcal{L}_\infty}\\ &\quad+\|C(s)\frac{1}{c_0^{\mathrm{T}}H(s)}c_0^{\mathrm{T}}\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}\\ &< \gamma_{u_e},\end{aligned}$$

which proves (21), and concludes the proof. □

Remark 4 The error bounds between $x(t)$ and $x_{\mathrm{ref}}(t)$, and $u(t)$ and $u_{\mathrm{ref}}(t)$ are uniformly bounded, and can be arbitrarily small. To understand how these bounds can be used for ensuring response with desired specifications, consider the following ideal control signal $u_{\mathrm{des}}(t) = k_{\mathrm{g}}r(t) - \sigma(x,t)$, which leads to the desired system in (3) by canceling the uncertainties exactly. In the closed-loop reference system (13), the uncertainties term in the control law is low-pass filtered by $C(s)$. Thus, the closed-loop reference system has a different response as compared with the desired one (3). The response of $x(t)$ can be made as close as desired to (3) by increasing the bandwidth of the low-pass filter $C(s)$.

5 Simulation

In this section, two different simulation scenarios will be provided to demonstrate the performance of the $\mathcal{L}_1$ adaptive control with sliding-mode based adaptive law.

Scenario 1 (L1-SMC vs. SMC for nonlinear matched uncertainties) Consider the system dynamics in (1) with $A = \begin{bmatrix}0&1&0\\0&0&1\\-1&-2&-2\end{bmatrix}$, $b = [0\ 0\ 1]^{\mathrm{T}}$, $c = [1\ 0\ 0]$, and $\sigma(x,t) = \sin x_1 + \mathrm{e}^{-|x_2|} + \arctan x_3 + \sin t$. Note that $\hat{\sigma}(x,t)$ is a nonlinear function of states and time, representing general nonlinear matched uncertainties. The control objective is to make the output track the desired system output in (3), and make the rest of states stable (SMC: sliding model control; L1-SMC: $\mathcal{L}_1$ adaptive control with sliding-mode based adaptive law).

For the L1-SMC design, switching function and $\rho(\tilde{x},t)$ in adaptive law are chosen as $s(\tilde{x}) = 5\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$ and $\rho(\tilde{x},t) = 2\|\tilde{x}\| + 10$, and the low-pass filter in the control law is chosen as $C(s) = \dfrac{50}{s+50}$. For the SMC design, switching function and $\rho(x,t)$ use the same parameters as the ones using in the L1-SMC design. In order to achieve reference tracking, a reference input filter [14,15], $\bar{r} = k_{\mathrm{g}}r$ with k_{g} defined in equation (3), is introduced for the SMC design.

The closed-loop response of L1-SMC and SMC are shown in Figs. 1–6, respectively. As shown in Figs. 1 and 5, the output $y(t)$ is coincide with the desire system's output $y_{\mathrm{des}}(t)$ under L1-SMC and SMC, which demonstrates good tracking performance. Yet, SMC generates an oscillating control signal (shown in Fig. 6) to achieve tracking, while L1-SMC generates a smooth control signal (shown in Fig. 4). The smooth control signal is due to the low-pass filtering design and sliding-mode based virtual control. For L1-SMC, all states $x(t)$ in Fig. 2 are stabilized and the estimation errors $\tilde{x}(t)$ in Fig. 3 have very small values. The estimation errors $\tilde{x}$ represent errors between the state predictor in equation (4) and plant (1). The small estimation errors demonstrate good estimation by using sliding-mode based adaptive law.

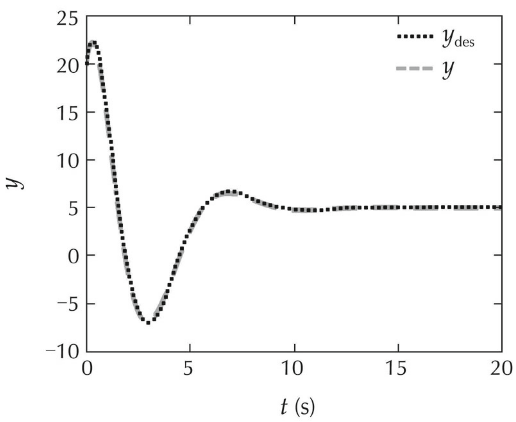

Fig. 1 Scenario 1: Output of the closed-loop system with L1-SMC.

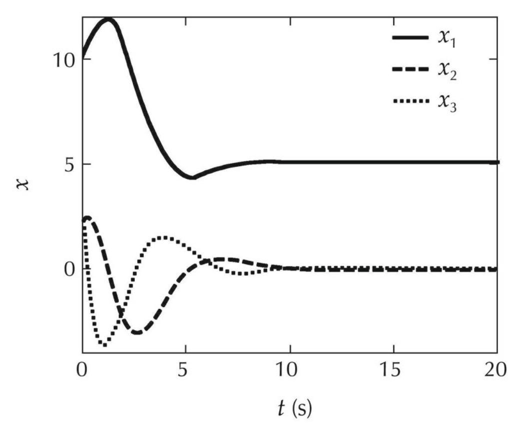

Fig. 2 Scenario 1: States of the closed-loop system with L1-SMC.

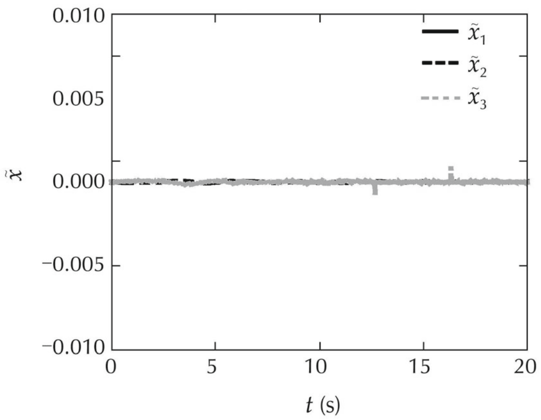

Fig. 3 Scenario 1: Estimation errors between state predictor and plant with L1-SMC.

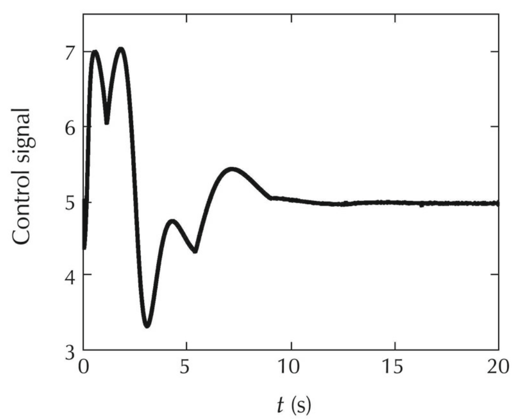

Fig. 4 Scenario 1: Control signal of the closed-loop system with L1-SMC.

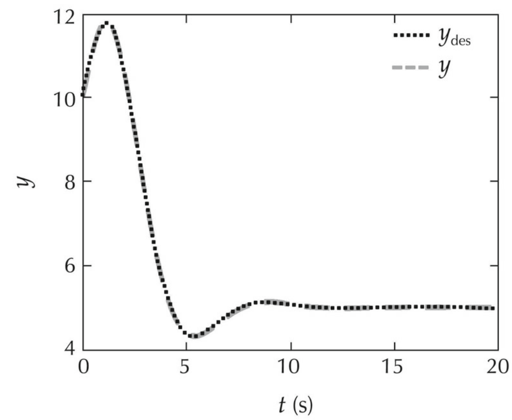

Fig. 5 Scenario 1: Output of the closed-loop system with SMC.

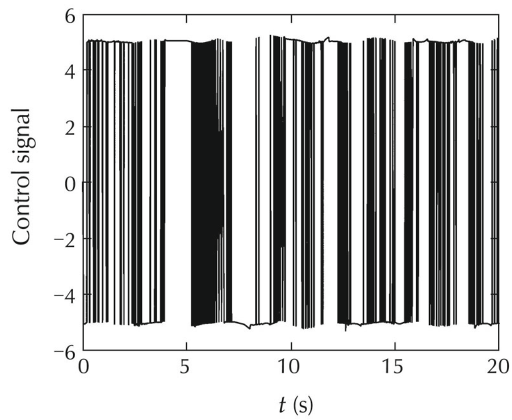

Fig. 6 Scenario 1: Control signal of the closed-loop system with SMC.

Scenario 2 (L1-SMC for systems with nonlinear dynamics and stable zero dynamics) Consider a nonlinear system with stable zero dynamics as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin x_1 & \mathrm{e}^{-|x_2|} & \arctan x_3 \end{bmatrix} x + bu, \quad y = cx,$$

where $b = [0\ 0\ 1]^{\mathrm{T}}$ and $c = [1\ 2\ 0]$. As stated in Remark 1, this type of nonlinear system can always be transformed into the dynamic system in equation (1) with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix}$ and $\sigma(x,t) = x_1 + (\sin x_1)x_1 + 2x_2 + \mathrm{e}^{-|x_2|}x_2 + 2x_3 + (\arctan x_3)x_3$.

For the L1-SMC design, switching function and $\rho(\tilde{x}, t)$ in adaptive law are chosen as $s(\tilde{x}) = 5\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$ and $\rho(\tilde{x}, t) = 5\|\tilde{x}\| + 10$, and the low-pass filter in the control law is chosen as $C(s) = 50/(s + 50)$.

The simulation results of L1-SMC for the nonlinear system with stable zero dynamics are shown in Figs. 7–10. The output $y(t)$ is coincide with the desire system's output $y_{\rm des}(t)$ in Fig. 7, which demonstrates good tracking performance. At the same time, all states $x(t)$ are stabilized as shown in Fig. 8.

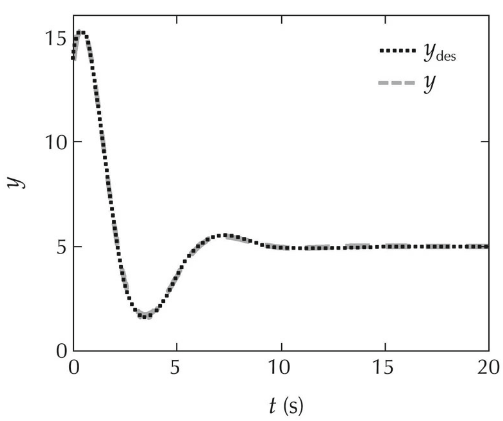

Fig. 7 Scenario 2: Output of the closed-loop system with L1-SMC.

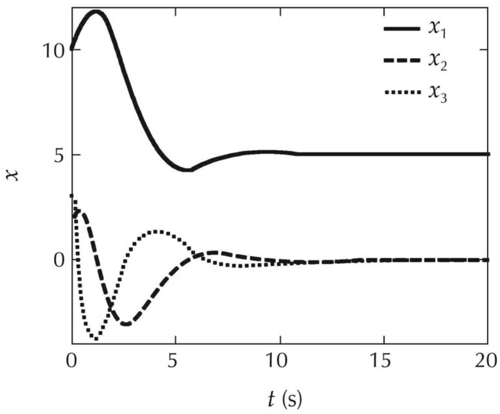

Fig. 8 Scenario 2: States of the closed-loop system with L1-SMC.

The estimation errors $\tilde{x}$ as shown in Fig. 9 have very small values, which demonstrates good estimation by using sliding-mode based adaptive law. In Fig. 10, the control signal is smooth.

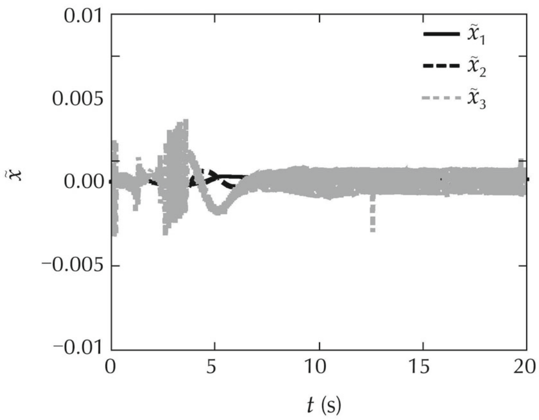

Fig. 9 Scenario 2: Estimation errors between state predictor and plant with L1-SMC.

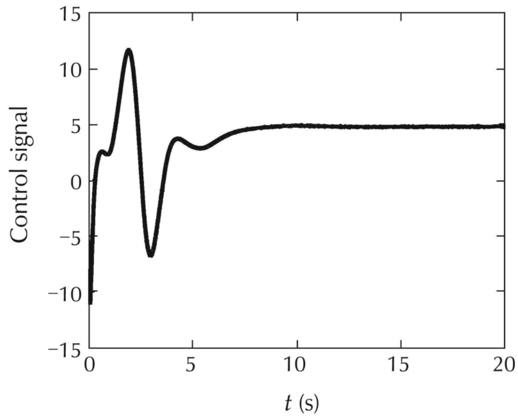

Fig. 10 Scenario 2: Control signal of the closed-loop system with L1-SMC.

6 Conclusions

In this paper, an adaptive control scheme with sliding-mode based adaptive law is presented. The robustness and control accuracy of sliding mode control are preserved while chattering effect is not introduced. The traditional sliding mode control is used as adaptive law in the $\mathcal{L}_1$ adaptive control architecture, which is a virtual control of error dynamics between estimated states and real states. The error bounds between estimated states and real states are bounded which can be arbitrarily small. The low pass filtering design in the control law prevents discontinuous signal in the adaptive law from appearing in the actual control signal while still maintaining control accuracy. The norm bounds of both states and control signals between the closed-loop adaptive system and the closed-loop reference system are characterized. Numerical simulation results demonstrate the performance of the presented adaptive control scheme.

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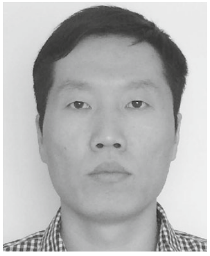

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