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Optimal control of quantum systems with SU(1, 1) dynamical symmetry

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Abstract

SU(1, 1) dynamical symmetry is of fundamental importance in analyzing unbounded quantum systems in theoretical and applied physics. In this paper, we study the control of generalized coherent states associated with quantum systems with SU(1, 1) dynamical symmetry. Based on a pseudo Riemannian metric on the SU(1, 1) group, we obtain necessary conditions for minimizing the field fluence of controls that steer the system to the desired final state. Further analyses show that the candidate optimal control solutions can be classified into normal and abnormal extremals. The abnormal extremals can only be constant functions when the control Hamiltonian is non-parabolic, while the normal extremals can be expressed by Weierstrass elliptic functions according to the parabolicity of the control Hamiltonian. As a special case, the optimal control solution that maximally squeezes a generalized coherent state is a sinusoidal field, which is consistent with what is used in the laboratory.

Keywords: Quantum control, optimal control, dynamical symmetry

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1 Introduction

In theoretical physics, the SU(1, 1) dynamical symmetry is of fundamental importance in algebraically solving Schrödinger equations with various potential functions (e.g., Morse potential, Pöschl-Teller potential, and Coulomb potential) [1–5], as well as in solid-state [6] and optical systems [7]. As one of the simplest Lie groups, SU(1, 1) group can be taken as an analytic continuation of SU(2) group and the Heisenberg group H(1) [8], but its noncompactness makes it more useful in generating unbounded and continuous energy spectrum in infinite-dimensional quantum mechanical systems [8–10].

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Analogous of the role of SU(2) dynamical symmetry in quantum information science, quantum systems with SU(1, 1) dynamical symmetry are basic building blocks in advanced quantum technologies [11], e.g., quantum squeezed lights and nanoscale mechanical resonators [12, 13], molecular disassociation/association control [14], high harmonic generation and continuous-variable quantum information [15, 16]. These applications lead to many control problems in engineering desired state or gate operations [14], which are generally realized by manipulating an electromagnetic field that directly or indirectly interacts with the system. Incorporated with the underlying physical disciplines, modern control theory [17] was recently introduced to design controls used in quantum chemistry [14, 18] and quantum information [16, 19]. In particular, the optimal control theory based on variational analysis (later extended to the so-called Maximum Principle to treat more general constraints) have been widely explored in quantum systems with both numerical and experimental applications.

In this paper, we study the minimum-fluence control problem for systems with SU(1, 1) dynamical symmetry, which is motivated by the common limitations on available control resources. For example, dynamical processes in nonlinear optical systems can only experience a very short period of time for effective interaction (e.g., in an optical crystal or an optical fiber), within which the control field has to be sufficiently strong to achieve the control goals. However, the power of realistic control resources is often limited; even if this is not a severe restriction, strong controls may introduce unwanted noises via excitation of environmental modes. Therefore, it is demanding to efficiently exploit the limited control bandwidth (jointly determined by available control power and pulse length) to achieve control goals. When the interaction time is fixed (e.g., in an optical fiber with fixed length), the use of control bandwidth can be optimized by minimizing the control fluence. To the authors' knowledge, there are some relevant studies on classical mechanical systems evolving on the SU(1, 1) group (e.g., non-Euclidean elastica [20–24]), but the applications to quantum systems have been rarely seen in the literature.

The balance of this paper is organized as follows. In Section 2, we introduce the model of quantum control systems with SU(1, 1) dynamical symmetry. In Sections 3 and 4, we derive the conditions for minimum-fluence control for systems with one or two controls, based on which the properties of the candidate optimal control solutions are discussed. Finally, conclusions are drawn in Section 5.

2 Quantum control systems with SU(1, 1) dynamical symmetry

Consider a quantum control system whose state is in a Hilbert space $\mathcal{H}$. When the field can be approximated as a classical field (e.g., an optical pulse that contains a sufficient large number of photons), we can describe the system with the following semiclassical model [25]:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \frac{1}{i\hbar}\hat{H}[\varepsilon(t)]\,|\Psi(t)\rangle, \quad |\Psi(0)\rangle = |\Psi_0\rangle, \tag{1}$$

where the time-dependent state $|\Psi(t)\rangle \in \mathcal{H}$ is governed by the Hamiltonian involving the control field $\varepsilon(t)$.

The quantum control system (1) is said to possess SU(1, 1) dynamical symmetry if the Lie algebra generated by the set of controlled Hamiltonians $\hat{H}[\varepsilon(t)]$ for all $\varepsilon(t)$ is isomorphic to su(1, 1) (see the following discussion for details), thereby generate a Lie semigroup of unitary propagators contained in SU(1, 1). Next, we will provide definitions of SU(1, 1) group, its Lie algebra and the corresponding generalized SU(1, 1) coherent states. To facilitate the following optimal control studies, the two control systems will be introduced in the context of quantum optics.

2.1 SU(1, 1) Lie group and SU(1, 1) coherent-state representation

The real Lie groups SU(2) and SU(1, 1) can be uniformly defined [20] as the set of all 2×2 matrices that satisfy $X^{\dagger}PX = P$, where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \tag{2}$$

with $\epsilon = 1$ for SU(2) and $\epsilon = -1$ for SU(1, 1), respectively. Correspondingly, the Lie algebras of these two groups consists of 2×2 skew P-Hermitian matrices $A^P = -A$, where the P-adjoint of X is defined as $X^P = PX^{\dagger}P$. Both Lie algebras of SU(1, 1) and SU(2) are three dimensional [26], and the elements in the Lie algebra satisfy $X^P = -X$, and the infinitesimal generators satisfy the following commutation relationships:

$$[K_0\ \ K_1] = K_2, \quad [K_1\ \ K_2] = \epsilon K_0, \quad [K_2\ \ K_0] = K_1. \tag{3}$$

A standard realization of the above basis is as follows:

$$K_0 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{4}$$

$$K_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (5)$$

$$K_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (6)$$

Although being defined in a unified manner, SU(2) and SU(1, 1) are essentially distinct in that SU(2) is a compact group but SU(1, 1) is noncompact. This can be seen from the associated quadratic form

$$\langle x, y\rangle_P = x^\dagger P y \quad (7)$$

on $\mathbb{C}^2$ induced by the P-adjoint. For the elements A and B in the Lie algebras su(1, 1), we have

$$\langle A, B\rangle_P = 2\mathrm{Tr}(AB^P). \quad (8)$$

The quadratic form for SU(2) is the same as the standard matrix inner products. However, the quadratic form for SU(1, 1) is not positive definite and thus is not an inner product, which is due to the noncompactness of SU(1, 1). Nevertheless, their invariance under the SU(1, 1) transformations is very useful in the following studies. For convenience, we call them pseudo inner products, many of whose calculations are parallel with those for SU(2). For example, if X, Y and Z are skew P-Hermitian, then

$$\langle X, Y\rangle_P = \langle Y, X\rangle_P^*, \quad (9)$$

$$\langle [X, Y], Z\rangle_P = \langle [Y, Z], X\rangle_P. \quad (10)$$

Based on the pseudo inner product defined on su(1, 1), one can further classify its elements according to their geometry. A matrix $M \in$ su(1, 1) and its exponential e^M are said to be elliptic (hyperbolic, parabolic) if $\langle M, M\rangle_P$ is positive (negative, zero). For examples, in the basis (4), K_0 is elliptic, while K_1 and K_2 are hyperbolic.

For quantum systems possessing SU(1, 1) dynamics symmetry, the evolution operator acting on the Hilbert space $\mathcal{H}$ forms a unitary representation of elements in su(1, 1). Due to the noncompactness of SU(1, 1), the Hilbert space $\mathcal{H}$ carrying such representations must be infinite dimensional. This feature makes essential distinctions between SU(1, 1) and SU(2) in that SU(1, 1) can serve to generate energy spectrum of an unbounded system but SU(2) can only be applied to bounded systems.

The unitary representation also induces an isomorphic transformation, say $\mathscr{R}$, from the Hamiltonian operator $\hat{H}$ to a skew P-Hermitian matrix in su(1, 1). It is easy to verify that the following equalities hold for arbitrary $X, Y \in$ su(1, 1):

$$\mathscr{R}([X, Y]) = [\mathscr{R}(X), \mathscr{R}(Y)], \quad (11)$$

$$\mathscr{R}(X^P) = \mathscr{R}(X)^\dagger. \quad (12)$$

In this paper, we are concerned with the control of an important class of quantum states, namely the SU(1, 1) generalized coherent states (GCS), which are generalized from the coherent states of a quantum harmonic oscillator. According to the standard Perelomov's definition [27], an SU(1, 1) GCS is some quantum state that can be generated by some SU(1, 1) transformation in the following way:

$$\begin{aligned} |\alpha\rangle &= \exp(\alpha\hat{K}_+ - \alpha^*\hat{K}_-)|0\rangle \\ &= (1-|\xi|^2)^k \sum_{n=0}^{\infty}\Big(\frac{\Gamma(m+2k)}{m!\Gamma(m)}\Big)^{\frac{1}{2}}\xi^n|n\rangle, \end{aligned} \quad (13)$$

where $\hat{K}_\pm = \mathscr{R}(K_1 \pm \mathrm{i}K_2)$ is a given unitary representation of su(1, 1); and $\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\}$ are the orthonormal basis of $\mathcal{H}$ with $|0\rangle$ being an arbitrary state in the Hilbert space (e.g., the vacuum state). The integer k is some constant that labels the specific Hilbert space. The set of GCSs generated from a fixed state $|0\rangle$ is invariant under SU(1, 1) transformations, and they form a two-dimensional submanifold that is isomorphic to the coset space SU(1, 1)/U(1), where U(1) is the one dimensional Lie group generated by K_0.

2.2 Quantum optical control systems with SU(1, 1) dynamical symmetry

The Hamiltonian of the control system (1) generally possesses the following form [28, 29]:

$$\hat{H}[\varepsilon(t)] = \hat{H}_0 + \varepsilon(t)\hat{H}_1, \quad (14)$$

where $\hat{H}_0$ and $\hat{H}_1$ are the internal and control Hamiltonians. The classical control field is a scalar real time-dependent function with tunable amplitude. In atomic or molecular control, $\hat{H}_1$ represents the electric (or magnetic) dipole that is coupled to an electric (or magnetic) field. Particularly, for the manipulation of quantum optical modes in a cavity or some dielectric medium, the control is realized by a pump field.

For example, two-photon processes are used for single mode squeezing with free and control Hamiltonians being [30–32]:

$$\hat{H}_0 = \hat{a}^\dagger\hat{a}, \quad \hat{H}_1 = \hat{a}^2, \quad (15)$$

where the annihilation operator $\hat{a}$ of the optical mode to be squeezed is defined on the Fock space spanned by photon number states $\{|0\rangle, |1\rangle, \ldots\}$. Another circumstance is applying the degenerate parametric down conversion corresponding to the following two-mode squeezing:

$$\hat{H}_0 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2, \quad \hat{H}_1 = \hat{a}_1 \hat{a}_2, \tag{16}$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators of the two created photons. In the two examples, the involved sets of operators satisfy the commutation relationship (3) when taking

$$\hat{K}_0 = \hat{a}^\dagger a, \quad \hat{K}_1 = \hat{a}^{\dagger 2} + \hat{a}^2, \quad \hat{K}_2 = i(\hat{a}^{\dagger 2} - \hat{a}^2), \tag{17}$$

or

$$\hat{K}_0 = \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2, \tag{18}$$
$$\hat{K}_1 = \hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2, \tag{19}$$
$$\hat{K}_2 = i(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2). \tag{20}$$

Systems (15) and (16) describe optical systems coupled with a linearly polarized field. When the field is circularly polarized, the Hamiltonian has the following form:

$$\hat{H}[\varepsilon(t)] = \hat{H}_0 + \varepsilon^*(t)\hat{H}_1 + \varepsilon(t)\hat{H}_1^\dagger, \tag{21}$$

where the value of the control field parameter $\varepsilon(t)$ is a complex number. Quantum control systems with one or two controls are to be studied in the following sections.

3 Minimum-fluence coherent state transfer with single control

As mentioned above, quantum control systems with SU(1,1) dynamical symmetry evolve on a finite dimensional manifold of generalized coherent states in the (infinite-dimensional) Hilbert space. Using the unitary representation $\mathscr{R}$ defined above, we can map the quantum state $|\Psi(t)\rangle$ to a two-dimensional complex vector in $\mathbb{C}^2$:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \mathscr{R}^{-1}(|\Psi(t)\rangle), \tag{22}$$

where $[1 \ \ 0]^{\mathrm{T}}$ corresponds to the root state $|0\rangle$ in (13). Correspondingly, the Hamiltonian $H[\varepsilon(t)]$ is mapped to a 2×2 skew P-Hermitian matrix

$$A[\varepsilon(t)] = \mathscr{R}^{-1}(\hat{H}[\varepsilon(t)]). \tag{23}$$

It is obvious that a control field $\varepsilon(t)$ drives system (1) from $|\Psi_0\rangle$ to $|\Psi_{\mathrm{f}}\rangle$ if and only if the same control drives the following system:

$$\dot{x}(t) = A[\varepsilon(t)]x(t), \quad x(0) = x_0 \in \mathbb{C}^2, \tag{24}$$

from $x_0 = \mathscr{R}^{-1}(|\Psi_0\rangle)$ to $x_{\mathrm{f}} = \mathscr{R}^{-1}(|\Psi_{\mathrm{f}}\rangle)$. Thus, it is more convenient to study optimal control problems with this finite-dimensional system.

The field fluence to be minimized is defined by the following cost functional:

$$J[\varepsilon(\cdot)] = \int_0^T |\varepsilon(t)|^2 \mathrm{d}t, \tag{25}$$

over the set of controls that are L^2-integrable over a fixed time period $[0, T]$. The Schrödinger equation (or the isomorphic one (24)) forms a dynamical constraint for the optimization.

Next, we apply the famous Pontrygin Maximum Principle [33, 34] to study the minimum-fluence control problem for systems with SU(1,1) dynamical symmetry. Consider a quantum system controlled by a single field, which by the isomorphic transformation $\mathscr{R}$ defined above, is translated to a two-dimensional optimal control system problem as follows:

$$\min_{\varepsilon(t)} \ J(\varepsilon) = \int_0^T |\varepsilon(t)|^2 \mathrm{d}t, \tag{26}$$
$$\text{s.t. } 1) \ \dot{x}(t) = [A_0 + \varepsilon(t)A_1]x(t), \quad t \in [0, T], \tag{27}$$
$$2) \ x(0) = x_0, \quad x(T) = x_{\mathrm{f}}, \tag{28}$$

where both A_0 and A_1 are skew P-Hermitian. The initial $x(0) = x_0$ and boundary condition $x(T) = x_{\mathrm{f}}$ are fixed, whereas T is not fixed.

First, we define the following pseudo Hamiltonian:

$$\boldsymbol{H}(t) = \frac{\lambda_0}{2}|\varepsilon(t)|^2 + \tilde{\lambda}^\dagger(t)[A_0 + \varepsilon(t)A_1]x(t), \tag{29}$$

where $\lambda_0 = 0$ or 1. The first term corresponds to the cost functional to be minimized, and the second term to the dynamical constraint (26) with a Lagrangian multiplier $\tilde{\lambda}(t)$. By defining $\lambda(t) = P\tilde{\lambda}(t)$, we have

$$\boldsymbol{H}(t) = \frac{\lambda_0}{2}|\varepsilon(t)|^2 + \langle \lambda(t), [A_0 + \varepsilon(t)A_1]x(t)\rangle_P. \tag{30}$$

The evolution of the Lagrangian multiplier is derived as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) &= -\frac{\partial \boldsymbol{H}(t)}{\partial x(t)} = -[A_0 + \varepsilon(t)A_1]^P \lambda(t) \\ &= [A_0 + \varepsilon(t)A_1]\lambda(t),\end{aligned} \tag{31}$$

whose boundary conditions are to be determined by the initial and boundary conditions in equation (28).

According to the Maximum Principle [34,35], any optimal control $\varepsilon(t)$ that minimizes the cost functional (25) must be an extremal curve that is defined as a control function satisfying

$$\frac{\partial \boldsymbol{H}(t)}{\partial \varepsilon(t)} = \lambda_0 \varepsilon(t) + \langle \bar{\lambda}(t), A_1 \bar{x}(t) \rangle_P \equiv 0, \quad \forall t \in [0, T], \tag{32}$$

where the extremal trajectory $(\bar{x}(t), \bar{\lambda}(t))$ is generated by the candidate optimal control $\varepsilon(t)$. The extremals corresponding to $\lambda_0 = 1$ are called normal, otherwise the extremals corresponding to $\lambda = 0$ are called abnormal [19].

To further determine the optimal control from the condition (32), we introduce the following auxiliary variables:

$$\nu_i(t) = -\langle \bar{\lambda}(t), A_i \bar{x}(t) \rangle_P, \quad i = 0, 1, 2, \tag{33}$$

where $A_2 = [A_0 \ \ A_1]$. By differentiating these variables in time, we obtain (a.e.)

$$\dot{\nu}_0 = \varepsilon(t)\nu_2, \tag{34}$$

$$\dot{\nu}_1 = -\nu_2, \tag{35}$$

$$\dot{\nu}_2 = -a_{01}\nu_0 + a_{00}\nu_1 + \varepsilon(t)(-a_{11}\nu_0 + a_{01}\nu_1), \tag{36}$$

where the structural constants a_{00}, a_{01} and a_{11} are defined as

$$a_{01} = \langle A_0, A_1 \rangle_P, \tag{37}$$

$$a_{00} = \langle A_0, A_0 \rangle_P, \tag{38}$$

$$a_{11} = \langle A_1, A_1 \rangle_P. \tag{39}$$

Equations (34)–(36) have a first integral

$$C = a_{11}\nu_0^2 - 2a_{01}\nu_0\nu_1 + a_{00}\nu_1^2 + \nu_2^2, \tag{40}$$

which is constant during the evolution and thereby restricts the trajectory $(\nu_0(t), \nu_1(t), \nu_2(t))$ on a two-dimensional surface $C = \text{const}$. This surface can be an ellipsoid, a hyperboloid, a paraboloid or a cone, which corresponds to the cases that A_2 is elliptic, hyperbolic, parabolic or conic, respectively. Next, we will use equations (34)–(36) to analyze the properties of abnormal and normal extremals.

3.1 Abnormal extremals

Abnormal extremals are defined as control solutions that satisfy (32) for $\lambda_0 = 0$, which correspond to the case that the cost is overwhelmed by the dynamical constraint. We have the following conclusions.

Theorem 1 If A_1 is non-parabolic, then there exists a unique abnormal extremal $\varepsilon(t) = -a_{01}a_{11}^{-1}$, a.e.

Proof By the definition of abnormal extremals, i.e., $\lambda_0 = 0$, condition (32) becomes

$$\frac{\partial \boldsymbol{H}(t)}{\partial \varepsilon(t)} = -\nu_1(t) \equiv 0, \quad \forall t \in [0, T], \tag{41}$$

which also implies that $\nu_0(t)$ is constant because the pseudo Hamiltonian $\boldsymbol{H}(t) = \nu_0(t) + \varepsilon(t)\nu_1(t)$ is time-invariant during the evolution. Further, equation (41) can be differentiated to yield $\nu_2(t) \equiv 0$.

Differentiating again $\nu_2(t) \equiv 0$, we have

$$\begin{aligned} &[-a_{01}\nu_0(t) + a_{00}\nu_1(t)] + \varepsilon(t)[-a_{11}\nu_0(t) + a_{01}\nu_1(t)] \\ &= -a_{01}\nu_0(t) - a_{11}\nu_0(t)\varepsilon(t) \equiv 0, \quad \text{a.e.}, \end{aligned} \tag{42}$$

from which it is easy to see that, when $a_{11} \neq 0$ (i.e., A_1 is non-parabolic), the abnormal control is a constant $\varepsilon(t) = -a_{01}a_{11}^{-1}$ (a.e.). □

3.2 Normal extremals

The normal extremals correspond to $\lambda_0 = 1$. With respect to such extremals, the second-order derivative of the pseudo Hamiltonian reads

$$\frac{\partial^2 \boldsymbol{H}(t)}{\partial \varepsilon(t)^2} = \lambda_0 = 1, \tag{43}$$

which implies that the corresponding control is always a local minimum. The necessary condition for candidate optimal controls can be expressed as $\varepsilon(t) = \nu_1(t)$, and thus we have (a.e.)

$$\dot{\nu}_0 = \nu_1\nu_2, \tag{44}$$

$$\dot{\nu}_1 = -\nu_2, \tag{45}$$

$$\dot{\nu}_2 = a_{01}\nu_1^2 - a_{01}\nu_0 + a_{00}\nu_1 - a_{11}\nu_0\nu_1. \tag{46}$$

In addition to the first integral (40), we have found that these equations possess another first integral $E = \nu_0 + \frac{\nu_1^2}{2}$. This quantity is called the elastic energy in [20].

The properties of normal extremals are essentially determined by the geometry of the parabolicity of A_1.

1) When the control Hamiltonian is non-parabolic, i.e., $a_{11} \neq 0$, we can define $\tilde{A}_0 = A_0 - a_{01}a_{11}^{-1}A_1$, $\tilde{A}_1 = A_1$ and $\tilde{\varepsilon}(t) = \varepsilon(t) + a_{01}a_{11}^{-1}$, with which equation (26) is

transformed to be

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = [\tilde{A}_0 + \tilde{\varepsilon}(t)\tilde{A}_1]x(t), \quad x(0) = x_0, \tag{47}$$

where one can examine that $\tilde{a}_{01} = \langle \tilde{A}_0, \tilde{A}_1 \rangle_P = 0$. Thus, without loss of generality, we can always assume that $a_{01} = 0$, under which the first integrals becomes

$$C = a_{11}\nu_0^2 + a_{00}\nu_1^2 + \nu_2^2, \quad E = \nu_0 + \frac{1}{2}\nu_1^2. \tag{48}$$

Then equations (44)–(46) can be reduced to a first-order differential equation

$$\dot{\nu}_1^2 = -\frac{a_{11}}{4}\nu_1^4 + (a_{11}E - a_{00})\nu_1^2 + (C - a_{11}E^2), \tag{49}$$

which, under the variable replacement $z = -\frac{a_{11}}{4}\nu_1^2 + \frac{a_{11}E - a_{00}}{3}$, leads to the following equation:

$$\dot{z}^2 = 4z^3 - g_2 z - g_3, \tag{50}$$

where g_2 and g_3 are functions of the constants E and C.

2) When the control Hamiltonian is parabolic, i.e., $a_{11} = 0$, we find that under the following transformation:

$$A_0 = \nu_0 + \frac{a_{00}}{3a_{01}}\tilde{A}_1 - \frac{2a_{00}^2}{9a_{01}^2}, \tag{51}$$

$$A_1 = \tilde{A}_1 - \frac{a_{00}}{3a_{01}}, \tag{52}$$

$$A_2 = \tilde{A}_2, \tag{53}$$

where we assume $a_{01} \neq 0$, $\tilde{a}_{00} = \langle \tilde{A}_0, \tilde{A}_0 \rangle_P = 0$. Hence, we can assume without loss of generality that $a_{00} = 0$ and consequently the first integrals become

$$C = -2a_{01}\nu_0\nu_1 + \nu_2^2, \quad E = \nu_0 + \frac{1}{2}\nu_1^2. \tag{54}$$

Similarly, let $\nu_1 = -4a_{01}^{-1}z$, equations (44)–(46) can be reduced to the same first-order differential equation

$$(\dot{z})^2 = 4z^3 - g_2 z - g_3, \tag{55}$$

where $g_2 = -\frac{a_{01}^2}{2}E$ and $g_3 = \frac{a_{01}^2}{16}C$.

The above analysis shows that the normal extremals can be solved from the same differential equation (50), whose solution belongs to the class of Weierstrass elliptic functions [36], and the parameters g_2 and g_3 can be determined by the boundary conditions $x(0) = x_0$, $x(T) = x_{\mathrm{f}}$ and the final time pseudo Hamiltonian condition $\boldsymbol{H}(T) = 0$. In particular, we can get more explicit solutions for the case that A_2 is parabolic. In [25], it was proven that A_2 is parabolic if and only if $a_{01}^2 - a_{00}a_{11} = 0$. Moreover, in this case, A_2 must be a linear combination of A_0 and A_1, causing that the Lie algebra generated by A_0 and A_1 is a proper Lie subalgebra of su(1, 1). When A_1 is non-parabolic, the reduced differential equation of ν_1 becomes

$$\dot{\nu}_1 = -\nu_0 = \frac{\nu_1^2}{2} - E, \tag{56}$$

whose solution is

$$\varepsilon(t) = \nu_1(t) = \begin{cases} -\sqrt{2E}\tanh(\sqrt{E}t + \gamma), & E > 0; \\ -\sqrt{-2E}\tan(\sqrt{-E}t + \gamma), & E < 0, \end{cases} \tag{57}$$

where the parameters E and γ are parameters to be determined from the boundary conditions. When A_2 is parabolic, the solution is simpler, i.e., $\varepsilon(t) = \gamma \mathrm{e}^{-t}$.

4 Optimal control of quantum squeezed states with two controls

Given a generalized SU(1, 1) coherent state written as $|\alpha = \theta \mathrm{e}^{-\mathrm{i}\phi}\rangle$, the parameter $\xi = \tanh\theta \mathrm{e}^{-\mathrm{i}\phi}$ characterizes a squeezed optical state in quantum optics. The value of $|\xi| = \tanh\theta \in [0, 1]$ represents the ratio of squeezing. For standard coherent states (e.g., those of a harmonic oscillator), $|\xi| = 0$, while the limit $|\xi| \to 1$ corresponds to a highly squeezed state. In the representation (24), one can verify that $\xi = x_2/x_1$ where $x = [x_1\ \ x_2]^{\mathrm{T}} = \mathscr{R}^{-1}(|\alpha\rangle)$.

Consider the quantum system controlled by a circularly polarized field, which is mapped as in equations (26)–(28) to the following optimal control system:

$$\min_{\varepsilon(t)} J(\varepsilon) = \int_0^T |\varepsilon(t)|^2 \mathrm{d}t, \tag{58}$$

$$\text{s.t. } 1) \begin{cases} \dot{x}(t) = [A_0 + \varepsilon^*(t)A_1 + \varepsilon(t)A_1^P]x(t), \\ x(0) = x_0, \quad t \in [0, T], \end{cases} \tag{59}$$

$$2)\ \psi[x_1(T), x_2(T)] = \xi_0 x_1(T) - x_2(T) = 0, \tag{60}$$

where T is not fixed. ξ_0 is the target squeezing ratio and $\psi[x_1(T), x_2(T)]$ in equation (60) is the constraints of the system final state.

More specifically, we consider the often encountered case (e.g., the two examples discussed in Section 2) that $A_0 = \omega_0 K_0$ and $A_1 = (K_1 + \mathrm{i}K_2)/2$, where ω_0 is the frequency of the optical mode and the basis matrices K_0,

K_1 and K_2 are defined in (4). Denote $\varepsilon(t) = u_1(t)+\mathrm{i}u_2(t)$, where $u_1(t)$ and $u_2(t)$ are the real and imaginary parts of $\varepsilon(t)$. Then we have the pseudo Hamiltonian with respect to the minimum-fluence control as

$$\begin{aligned}\boldsymbol{H}(t) =& \frac{\lambda_0}{2}[u_1^2(t) + u_2^2(t)] \\ &+ \langle \lambda(t), [\omega_0 K_0 + u_1(t)K_1 + u_2(t)K_2]x(t)\rangle_P, \end{aligned} \tag{61}$$

where $\lambda_0 = 0$ or 1, and the evolution of the Lagrangian multiplier $\lambda(t)$ obeys the same differential equation of $x(t)$ because

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) &= -[\omega_0 K_0 + u_1(t)K_1 + u_2(t)K_2]^P \lambda(t) \\ &= [\omega_0 K_0 + u_1(t)K_1 + u_2(t)K_2]\lambda(t). \end{aligned} \tag{62}$$

Let $\nu_i(t) = -\langle \lambda(t), K_i x(t)\rangle_P$, $i = 0, 1, 2$. Their evolution obeys the following equations:

$$\dot{\nu}_0(t) = u_1(t)\nu_2(t) - u_2(t)\nu_1(t), \tag{63}$$

$$\dot{\nu}_1(t) = -\omega_0\nu_2(t) - u_2(t)\nu_0(t), \tag{64}$$

$$\dot{\nu}_2(t) = \omega_0\nu_1(t) + u_2(t)\nu_0(t). \tag{65}$$

Similarly, one can prove that the abnormal extremals (i.e., $\lambda_0 = 0$) can only be zero control $\varepsilon(t) \equiv 0$, which is trivial in practice.

For normal extremals (i.e., $\lambda_0 = 1$), the Pontryagin Maximum Principle reads

$$\frac{\partial \boldsymbol{H}}{\partial u_i(t)} = u_i(t) - \nu_i(t) \equiv 0, \quad i = 1, 2. \tag{66}$$

Combined with equation (63), it can be seen that

$$\dot{\nu}_0(t) = 0, \tag{67}$$

implying that $\nu_0(t) \equiv \delta$ is constant. Thus, equations (64) and (65) become

$$\dot{\nu}_1(t) = -(\omega_0 + \delta)\nu_2(t), \tag{68}$$

$$\dot{\nu}_2(t) = (\omega_0 + \delta)\nu_1(t). \tag{69}$$

Therefore, both $\nu_1(t) = u_1(t)$ and $\nu_2(t) = u_2(t)$ are sinusoidal functions of time, where δ is the detuning frequency from the resonating frequency ω_0. The resulting control field can be written as $\varepsilon(t) = \nu_1(t) + \mathrm{i}\nu_2(t) = M\mathrm{e}^{\mathrm{i}[(\omega_0+\delta)t+\psi]}$. M and ψ can be determined from the boundary conditions

$$x(0) = x_0, \tag{70}$$

$$\lambda(T) = \frac{\partial \psi'}{\partial x(T)}\gamma(T), \tag{71}$$

$$\psi[x_1(T), x_2(T)] = 0, \tag{72}$$

and the final time pseudo Hamiltonian condition $H(T) = 0$.

The Schödinger equation (24) under such sinusoidal control functions is analytically solvable. Using the rotating-frame transformation $\bar{x}(t) = \mathrm{e}^{-[(\omega_0+\delta)t+\psi]K_0}x(t)$, which leads to

$$\dot{\bar{x}}(t) = (\delta K_0 + MK_1)\bar{x}(t), \tag{73}$$

and the solution is $\bar{x}(t) = \mathrm{e}^{(\delta K_0 + MK_1)t}\bar{x}(0)$. Then, after transforming back to $x(t)$, the final solution to the optimal trajectory can be derived to be

$$x(t) = \mathrm{e}^{[(\omega_0+\delta)t+\psi]K_0}\mathrm{e}^{(\delta K_0+MK_1)t}\mathrm{e}^{-\psi K_0}x(0), \tag{74}$$

where $t \in [0, T]$ and $x(0) = (1, 0)'$ is the vacuum state.

Therefore, for a normal extremal, the field fluence is $J = \frac{M^2T}{2}$ and the corresponding squeezing parameter can be computed as

$$\xi(T) = \frac{x_2(T)}{x_1(T)} = \begin{cases} \dfrac{\mathrm{i}M\mathrm{e}^{\mathrm{i}[(\omega_0+\delta)T+\psi]}}{\Omega \coth\frac{\Omega T}{2} - \mathrm{i}\delta}, & |\delta| \leqslant M, \\ \dfrac{\mathrm{i}M\mathrm{e}^{\mathrm{i}[(\omega_0+\delta)T+\psi]}}{\Omega \cot\frac{\Omega T}{2} - \mathrm{i}\delta}, & |\delta| \geqslant M, \end{cases} \tag{75}$$

where $\Omega = \sqrt{|M^2 - \delta^2|}$. Using the final state constraint equation (60), we take the absolute value of the above equation and give that

$$MT = \begin{cases} \dfrac{|\xi_0|}{\sqrt{1-|\xi_0|^2}}\Omega T \operatorname{csch}\dfrac{\Omega T}{2}, & |\delta| \leqslant M, \\ \dfrac{|\xi_0|}{\sqrt{1-|\xi_0|^2}}\Omega T \csc\dfrac{\Omega T}{2}, & |\delta| \geqslant M. \end{cases} \tag{76}$$

The dependence of M on the detuning parameter δ is depicted in Fig. 1, where the minimum value of the field amplitude (for fixed final time T) can be seen to be reached at $\delta = 0$ (i.e., on resonance), where

$$\sinh^2\frac{MT}{2} = \frac{|\xi_0|^2}{1-|\xi_0|^2}, \tag{77}$$

or equivalently,

$$\frac{MT}{2} = \operatorname{arctanh}|\xi_0|, \tag{78}$$

corresponding to

$$J^{\min} = \frac{M^2T}{2} = \frac{2}{T}(\operatorname{arctanh}|\xi_0|)^2. \tag{79}$$

The phase parameter ψ can be determined by matching the phase of ξ_0, i.e.,

$$\psi = \arg \xi(T) - \omega_0 T - \frac{\pi}{2} \pmod{2\pi}. \tag{80}$$

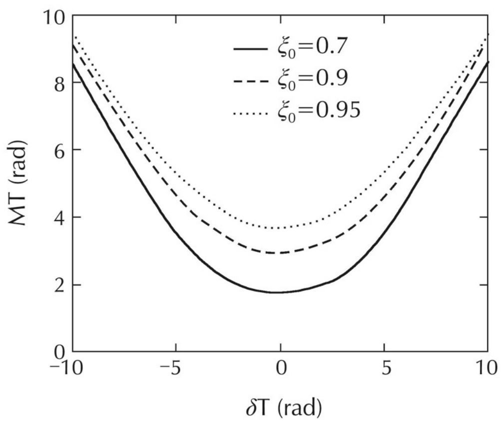

Fig. 1 The minimal amplitude M of the control field required for state squeezing at the ratio of ξ_0 under various detuning frequency δ, where T is the final time. The required field becomes stronger when the detuning is greater or the final state is more squeezed.

From the above derivation, one can also solve the dual problem of maximizing the squeezing ratio $\xi(T)$ under given control fluence $F = \frac{M^2 T}{2}$. Note that the restriction on the fluence implies that $M = \sqrt{2FT^{-1}}$, thus

$$|\xi(T)|^2 = \begin{cases} \dfrac{2F}{\Omega^2 T \operatorname{csch}^2(\frac{\Omega T}{2}) + 2F}, & |\delta| \leqslant M, \\ \dfrac{2F}{\Omega^2 T \csc^2(\frac{\Omega T}{2}) + 2F}, & |\delta| \geqslant M, \end{cases} \tag{81}$$

which, similarly, can be proven to be maximized at $\delta = 0$ and $\Omega = M = \sqrt{2FT^{-1}}$, which leads to

$$|\xi(T)|_{\max} = \tanh \sqrt{\frac{FT}{2}}. \tag{82}$$

From the above results, the state can be squeezed with a higher degree when increasing the quantity FT, which can be done by either increasing the fluence of the field or extending the time of interaction of the pump field with the medium.

5 Conclusions

The optimal control for quantum system has obtained many fruitful results [37, 38]. In this paper, we used the optimal control method to study the minimum-fluence control of quantum generalized coherent states in systems with SU(1, 1) dynamical symmetry. It was found that, according to the geometry of the system control Hamiltonian (i.e., its parabolicity), the optimal control solutions with a single control field can be expressed in terms of Weierstrass elliptic functions. Different from the related work on the group SU(2) [19], the properties of extremal controls are much more complicated due to the noncompactness of SU(1, 1) group. For systems with two controls, we proved that the optimal control is a monochromatic field that is resonant with the frequency of the mode to be controlled, which is exactly what is used in laboratory from physical intuition. These results reflect the complexity in controlling unbounded quantum systems even on a finite dimensional submanifold. In the future, the optimal control theory will be applied to quantum systems with more general dynamical symmetries. We will also make use of optimal control theory to study other issues, e.g., the decoherence problems [39, 40] in open SU(1, 1) dynamical symmetry.

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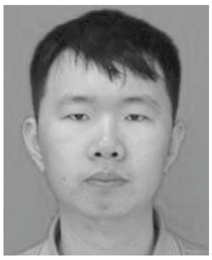

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