Control Theory Tech, Vol. 14, No. 3, pp. 250–259, August 2016

Control Theory and Technology

http://link.springer.com/journal/11768

Active disturbance rejection control: between the formulation in time and the understanding in frequency

Qing ZHENG [1†], Zhiqiang GAO [2]

1.*Department of Electrical and Computer Engineering, Gannon University, Erie, PA 16541, U.S.A.;*

2.*Center for Advanced Control Technologies, Department of Electrical and Computer Engineering, Cleveland State University, Cleveland, OH 44115, U.S.A.*

Received 14 April 2016; revised 8 June 2016; accepted 12 June 2016

Abstract

With the rapid deployments of the active disturbance rejection control (ADRC) as a bonafide industrial technology in the background, this paper summarizes some recent results in the analysis of linear ADRC and offers explanations in the frequency response language with which practicing engineers are familiar. Critical to this endeavor is the concept of bandwidth, which has been used in a more general sense. It is this concept that can serve as the link between the otherwise opaque state space formulation of the ADRC and the command design considerations and concerns shared by practicing engineers. The remarkable characteristics of a simple linear ADRC was first shown in the frequency domain, followed by the corresponding analysis in time domain, where the relationship between the tracking error and the ADRC bandwidth is established. It is shown that such insight is only possible by using the method of solving linear differential equations, instead of the more traditional techniques such as the Lyapunov methods, which tend to be more conservative and difficult to grasp by engineers. The insight obtained from such analysis is further demonstrated in the simulation validation.

Keywords: Active disturbance rejection control, extended state observer, uncertain systems, stability, bandwidth

DOI 10.1007/s11768-016-6059-9

1 Introduction

As the object to be controlled, physical plants in real world are not just nonlinear and time-varying but also highly uncertain. As the well-known control theorist Roger Brockett puts it: "If there is no uncertainty in the system, the control, or the environment, feedback control is largely unnecessary" [1]. For much of its history, however, mathematical control theory has been developed largely based on the premise that a physical plant behaves rather closely as its mathematical model describes. Serving as the point of departure in control

†Corresponding author.
E-mail: zheng003@gannon.edu. Tel: 1-814-871-5617; fax: 1-814-871-7617.

system design, this assumption does not reflect either the necessity of feedback control, nor the physical reality. The premise of model has stimulated in the last few decades lively debates and rapid new developments, such as those under the umbrella of robust, adaptive, and nonlinear control. But the dependence on the model proves difficult to shake loose even though the engineering practice of automatic control has taught us that PID, with over a hundred years of history, is still the king, and that engineers by and large have little use of design techniques premised on a detailed mathematical model of the physical process to be controlled.

The practicality of a model-based design can be problematic in two regards: 1) it could be rather expensive to obtain a detailed mathematical model; 2) even if such a model is obtained, the uncertainties in the process, particularly the changes in the system dynamics, could easily render such model obsolete during operation. Such problems prove to be hard to overcome even with the most advanced techniques such as those known as robust control, where the controller is made tolerant of uncertainties to some degree but still requires nonetheless a fairly detailed and accurate mathematical model of the plant. For example, the robust control design methodology based on the small gain theorem does allow a small amount of uncertainties in plant dynamics, but not anywhere near the magnitude often encountered in practice. The problem of controlling a process of a large amount of dynamic uncertainties remains unsolved until a new paradigm, namely active disturbance rejection control (ADRC), came to the scence.

The ADRC resonates with practical minded researchers from the very beginning when Han argued that there must be a way to control a process independent of its mathematical model. The framework and conceptual underpining gradually took shape in the span of two decades between 1989 and 2009, as explained in [2–11]. They fundamentally differ from other disturbance-centric design methods in the very concept of disturbance, which has been widely taken as forces external to the process to be controlled. Han inherited the notion of disturbance from H. S. Tsien that is more general and inclusive, including both the internal as well as the external disturbance. Tsien coined the term internal disturbance but Han took it to the next level, namely the total disturbance, which could very well be a function of the states of the process. In doing so, Han found a way to deal with the problem of large amount of dynamic uncertainties and to escape from the suffercating hold of the model-based design methodology.

The solution turns out to be a simple one: treat the process dynamics and external disturbance alike; lump them into a whole called the total disturbance; and then find a way to estimate and cancel it, reducing the process dynamics to an ideal, disturbance-free form. It is therefore obvious that whole enterprise comes down to the question of if such disturbance can be indeed estimated and cancelled. To this end Han left us with the extended state observer (ESO) [4], in a manner of experimental science: daring hypothesis followed by two decades of meticulously constructed tests, both in simulation and experimentations. In fact, Han pioneered the method of investigation in search of effective control mechanisms using computer simulation as the main tool [11].

To be sure, disturbance estimation and cancellation has been studied by many researchers over the years and many solutions have been offered, such as the unknown input observer (UIO) [12–19], the disturbance observer (DOB) [20–27], and the perturbation observer (POB) [28–31]. Two recent surveys can be found in [32, 33]. The key difference from ADRC is that they are intended to compliment the existing model-based paradigm rather than replacing with a new one. The new ADRC paradigm started slowly but picked up speed recently, largely propelled by its large scale adoption as a viable industrial solution, threatening the dominance of the PID solution. What was for a long time an experimental solution all the sudden acquired the attention of researchers intending to grasp its stability properties. This paper summarizes some recent results in the analysis of linear ADRC (LADRC) and offers explanations in the frequency response language with which practicing engineers are familiar.

The first attempt at the rigorous study of stability of the ADRC solution can be found in [34], where, for the sake of ease, the nonlinear gain structure of the original ADRC is replaced with a linear one. For the first time the convergence of the ESO and the bound on the tracking error in the ADRC were established, which lent support to the engineering success of the ADRC and further stimulated research interests on the subject. The research has grown more intensely and fruitfully since then, as can be seen in the more recent publications in [35–40]. But there is one nagging problem that refuses to go away: the more rigorous study of ADRC has done little to provide guidance to its engineering applications. This paper intends to address this issue, as one of the languages. In particular, we believe that the language of the time domain analysis based on solving differential equations must be intimately connected with the language of frequency responses with which engineers are familiar. This can be done, as shown earlier in [34], by solving the differential equation and examining the properties of the solutions directly, instead of using the Lyapunov type of methods that tend to be rather conservative and cubersome.

The paper is organized as follows. The time domain formulation of the ADRC is presented in Section 2. The engineering insight from frequency responses is discussed in Section 3. The time domain and frequency domain connection is given in Section 4. The time domain validation is shown in Section 5. The paper ends with a few concluding remarks in Section 6.

2 Time domain formulation of the ADRC

The ADRC was originally proposed as a combinatin of a tracking differentiator (TD) plus an ESO with a nonlinear form [3]. The key of the ADRC is the ESO. In [41], the ADRC was proposed to be realized with a PD controller and a linear extended state oberser (LESO), formulating a LADRC. In this paper, the presented ADRC approach refers to LADRC.

First the ESO design is presented. Consider a generally nonlinear time-varying dynamic system with single-input, u, and single-output y,

$$y^{(n)}(t) = f(y^{(n-1)}(t), y^{(n-2)}(t), \dots, y(t), w(t)) + bu(t), \quad (1)$$

where w is the external disturbance and b is a given constant. Here $f(y^{(n-1)}(t), y^{(n-2)}(t), \dots, y(t), w(t))$, or simply denoted as f, represents the nonlinear time-varying dynamics of the plant that is unknown. That is, for this plant, only the order and the parameter b are given. The ADRC is a unique method designed to tackle this problem. It is centered around estimation of, and compensation for, f. To this end, assuming f is differentiable and let $h = \dot{f}$, (1) can be written in an augmented state space form

$$\begin{cases} \dot{x}_1 = x_2, \\ \vdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = x_{n+1} + bu, \\ \dot{x}_{n+1} = h(x, u, w, \dot{w}), \\ y = x_1, \end{cases} \quad (2)$$

where $x = [x_1 \ x_2 \ \cdots \ x_{n+1}]^{\mathrm{T}} \in \mathbb{R}^{n+1}$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the state, input and output of the system, respectively. Any state observer of (2), will estimate the derivatives of y and f since the latter is now a state in the extended state model. Such observers are known as ESO.

In many real world scenarios, the plant dynamics represented by f is mostly unknown. The ESO design for a system with dynamics largely unknown is shown below.

With u and y as inputs, the ESO of (2) is given as

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + l_1(x_1 - \hat{x}_1), \\ \vdots \\ \dot{\hat{x}}_{n-1} = \hat{x}_n + l_{n-1}(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_n = \hat{x}_{n+1} + l_n(x_1 - \hat{x}_1) + bu, \\ \dot{\hat{x}}_{n+1} = l_{n+1}(x_1 - \hat{x}_1), \end{cases} \quad (3)$$

where $\hat{x} = [\hat{x}_1 \ \hat{x}_2 \ \cdots \ \hat{x}_{n+1}]^{\mathrm{T}} \in \mathbb{R}^{n+1}$, and l_i, $i = 1, 2, \dots, n+1$, are the observer gain parameters to be chosen. In particular, let us consider a special case where the gains are chosen as

$$[l_1 \ l_2 \ \cdots \ l_{n+1}] = [\omega_{\mathrm{o}}\alpha_1 \ \omega_{\mathrm{o}}^2\alpha_2 \ \cdots \ \omega_{\mathrm{o}}^{n+1}\alpha_{n+1}] \quad (4)$$

with $\omega_{\mathrm{o}} > 0$. Here $\alpha_i, i = 1, 2, \dots, n+1$, are selected such that the characteristic polynomial $s^{n+1} + \alpha_1 s^n + \dots + \alpha_n s + \alpha_{n+1}$ is Hurwitz. For simplicity, let $s^{n+1} + \alpha_1 s^n + \dots + \alpha_n s + \alpha_{n+1} = (s+1)^{n+1}$ where $\alpha_i = \dfrac{(n+1)!}{i!(n+1-i)!}$, $i = 1, 2, \dots, n+1$. Then the characteristic polynomial of (6) is

$$\begin{aligned} \lambda_{\mathrm{o}}(s) &= s^{n+1} + \omega_{\mathrm{o}}\alpha_1 s^n + \dots + \omega_{\mathrm{o}}^n \alpha_n s + \omega_{\mathrm{o}}^{n+1}\alpha_{n+1} \\ &= (s + \omega_{\mathrm{o}})^n. \end{aligned} \quad (5)$$

and ω_{o}, the observer bandwidth, becomes the only tuning parameter of the observer.

When a system model is known, then with the given function h, the ESO of (2) now takes the form of

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + l_1(x_1 - \hat{x}_1), \\ \vdots \\ \dot{\hat{x}}_{n-1} = \hat{x}_n + l_{n-1}(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_n = \hat{x}_{n+1} + l_n(x_1 - \hat{x}_1) + bu, \\ \dot{\hat{x}}_{n+1} = l_{n+1}(x_1 - \hat{x}_1) + h(\hat{x}, u, w, \dot{w}). \end{cases} \quad (6)$$

Assume that the control design objective is to make the output of the plant in (1) follow a given, bounded, reference signal r, whose derivatives, $\dot{r}, \ddot{r}, \dots, r^{(n)}$, are also bounded. Let $[r_1 \ r_2 \ \dots \ r_n \ r_{n+1}]^{\mathrm{T}} = [r \ \dot{r}_1 \ \cdots \ \dot{r}_{n-1} \ \dot{r}_n]^{\mathrm{T}}$.

Employing the ESO of (2) in the form of (3) or (6), the ADRC control law is given as

$$u = [k_1(r_1 - \hat{x}_1) + k_2(r_2 - \hat{x}_2) + \ldots + k_n(r_n - \hat{x}_n) \\ -\hat{x}_{n+1} + r_{n+1}]/b, \quad (7)$$

where $k_i, i = 1, 2, \ldots, n$, are the controller gain parameters selected to make $s^n + k_n s^{n-1} + \ldots + k_1$ Hurwitz. The closed-loop system becomes

$$y^{(n)}(t) =(f - \hat{x}_{n+1}) + k_1(r_1 - \hat{x}_1) + k_2(r_2 - \hat{x}_2) + \ldots \\ + k_n(r_n - \hat{x}_n) + r_{n+1}. \quad (8)$$

Note that with a well-designed ESO, the first term in the right hand side (RHS) of (8) is negligible and the rest of the terms in the RHS of (8) constitutes a generalized PD controller with a feedforward term. It generally works very well in applications but the issues to be addressed are: 1) the stability of the closed-loop system (8); and 2) the bound of the tracking error. Note that the separation principal does not apply here because of the first term in the RHS of (8).

3 Engineering insight from frequency responses

Most of the development and analysis of the ADRC have only been shown in time domain. In [42], frequency-domain analysis of the ADRC is performed to quantify its performance and stability characteristics. In [43], it is shown that the amount of uncertainties can be reduced by way of active disturbance rejection, implemented in an inner loop to produce a well-behaved plant, which is then regulated by another controller in the outer loop. In [44], the ESO is brought into the frequency domain to show to what degree it forces the plant to behave like cascaded integrators and what can be done to improve the performance when the ESO is bandwidth limited. Some rigorous analysis for the frequency domain properties of ADRC has been given in [45].

3.1 Frequency response analysis

Consider a linear time-invariant second-order plant:

$$\ddot{y} = -a_1\dot{y} - a_0 y + bu \quad (9)$$

with a_0 and a_1 unknown, $f = -a_1\dot{y} - a_0 y$ in this particular case. Since both the plant and the controller are linear, the robustness of the control system can be evaluated using frequency response. If ADRC indeed estimates f and cancels it out, then we should see very little change in bandwidth and stability margins when a_0 and a_1 vary.

The Bode plots of the loop gain transfer function are shown in Fig. 1. With $\omega_c = \omega_o = 100$ rad/s, $b = 206.25$, $a_1 = 3.085$, and $a_0 = [0 \ \ 0.1 \ \ 1 \ \ 10 \ \ 100]$, Fig. 1 (left) shows that, remarkably, gain margin, phase margin and cross-over frequency are almost immune to changes in a_0. Similarly, with $\omega_c = \omega_o = 100$ rad/s, $b = 206.25$, $a_0 = 0$, and $a_1 = [0.1 \ \ 1 \ \ 3.085 \ \ 10 \ \ 100]$, Fig. 1 (right) demonstrates that gain margin, phase margin and cross-over frequency are just as insensitive to changes in a_1 as to those in a_0.

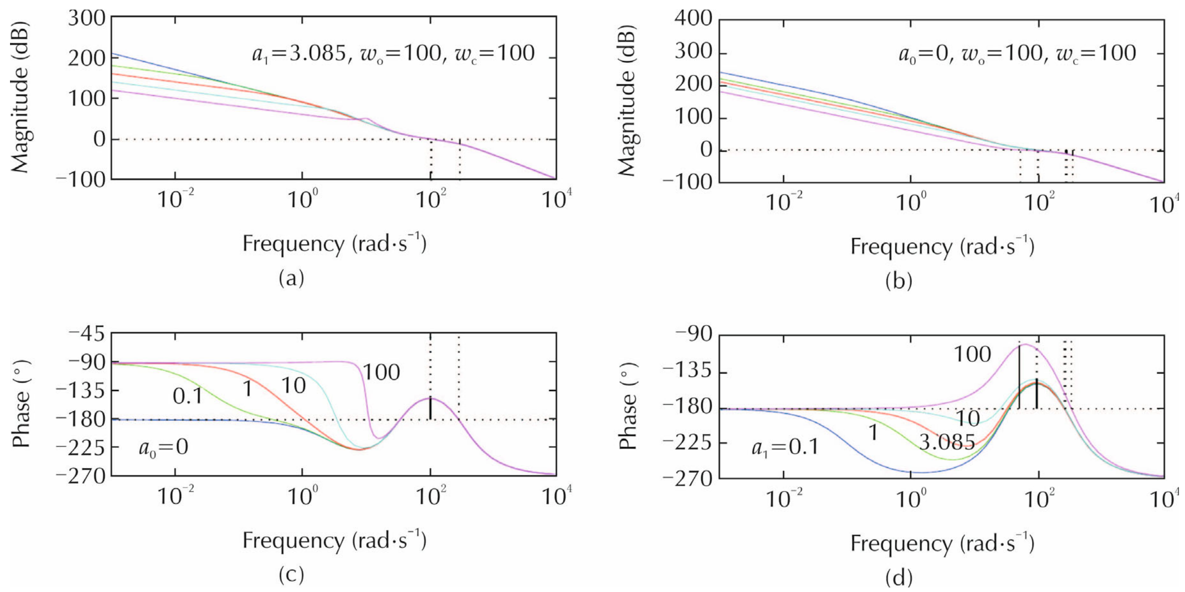


Fig. 1 Loop gain Bode plots at different a_0 and a_1. (a) and (c): $a_0 = 0, 0.1, 1, 10, 100$. (b) and (d): $a_1 = 0.1, 1, 3.085, 10, 100$.

The results show that the active disturbance rejection based control system possesses a level of robustness that is rarely seen. The bandwidth and stability margins, in particular, are kept almost unchanged as the plant parameters vary significantly.

3.2 Uncertainty reduction through active disturbance rejection

The theme of modern control is how to get around the unknowns, i.e., model uncertainties and disturbances, so that they do not degrade what is valued: stability and performance. In [43], it is demonstrated that the uncertainty stemming from both the external disturbance and the unknown internal dynamics, which is the subject of intense research efforts in the last few decades, can be greatly reduced through active disturbance rejection. Accordingly, it is demonstrated that control of uncertain system can be carried out in two steps: 1) reducing the uncertain plant, via active disturbance rejection, to a class of cascaded integral plants; and 2) design the front end controller for these compensated plants.

To demonstrate the effectiveness of the ESO in uncertainty reduction, consider a second order plant with a nominal transfer function: $G = \dfrac{200}{s(s+3)}$. The unknown dynamics is characterized in the form of $W_{\mathrm{ud}} = \dfrac{\tau s + r_0}{\frac{\tau}{r_\infty}s + 1}$, where r_0 is the modeling error in steady state, r_∞ is an uncertainty scalar at high frequency, and τ^{-1} is the frequency at which the system is completely unknown. Here $r_0 = 1$, $\tau^{-1} = 0.2\pi$, and $r_\infty = 5$. The perturbed plant is of the form:

$$G_{\mathrm{p}} = G(1 + W_{\mathrm{ud}}\Delta), \quad |\Delta| \leqslant 1. \tag{10}$$

If the ESO can fairly estimate the total disturbance, then the purtubed plant (10) can be reduced to $\ddot{y} = u_0$. Bode plots of the transfer function for the plant $\ddot{y} = u_0$ from u_0 to y are shown for different observer bandwidths in Fig. 2. It demonstrates the amount of uncertainty reduction by the ESO. Clearly, the quality of uncertainty reduction is directly correlated to the bandwidth: the higher the ω_{o}, the closer the compensated plant is to the ideal double integral plant. From Fig. 2 it is concluded that the plant from u_0 to y is reduced to a pure double integrator with very small error up to the frequency of $0.1\omega_{\mathrm{o}}$. That is, the control design problem is reduced to dealing with a pure double integral plant at or below the frequency of $0.1\omega_{\mathrm{o}}$.

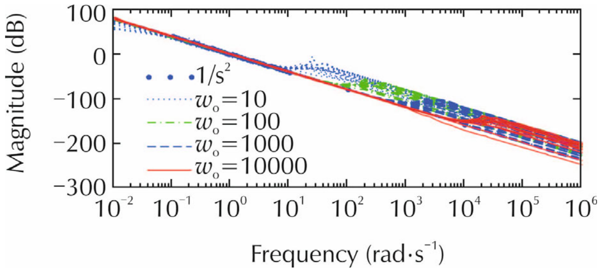


Fig. 2 Magnitude plot of the compensated plant.

3.3 The enhanced ADRC design with a low observer bandwidth

In [44], a more delicate case is considered when the bandwidth of ESO is quite limited, posing the question of how to best compensate for the lack of high bandwidth in the ESO. Such problems are approched from the frequency domain, assuming the plant is represented by a transfer function $G_{\mathrm{p}}(s)$. The ESO in the estimation of f, is shown in Fig. 3, where the ESO is respectively represented by the transfer functions from u to $\hat{f}$ and y to $\hat{f}$ (denoted as F_u and F_y, respectively).

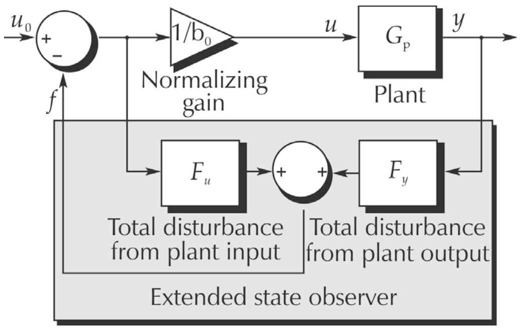


Fig. 3 Single integral plant acting as double integral.

By assigning all poles of the observer to ω_{o}, denoted as the observer bandwidth, the process of selecting gains in ESO becomes one of simply tuning ω_{o}. The modified plant, i.e., the transfer function from u_0 to y, can be shown as

$$\bar{G}_{\mathrm{p}}(s) = \frac{y(s)}{u_0(s)} = \frac{G_{\mathrm{p}}(s)b_0{}^{-1}}{1 + \dfrac{G_{\mathrm{p}}(s)b_0{}^{-1}s^n - 1}{(s/\omega_{\mathrm{o}} + 1)^{n+1}}}. \tag{11}$$

Note that the denominator contains a low-pass filter of order $n+1$ with a corner frequency of ω_{o}. If this is imagined as an ideal filter, where it acts as unity gain at and below the corner frequency but zero gain above it, the frequency response of (11) can be expressed as

$$\bar{G}_{\rm p}({\rm j}\omega) \approx \begin{cases} \dfrac{1}{({\rm j}\omega)^n}, & \omega \ll \omega_{\rm o}, \\ G_{\rm p}({\rm j}\omega)b_0^{-1}, & \omega \gg \omega_{\rm o}. \end{cases} \tag{12}$$

It can be seen from (12) that the modified plant acts as perfect integrators of order n within the bandwidth of the observer. At high frequencies, it will instead follow the response of the plant. It can be assumed that if an infinite bandwidth could be selected in an ideal world without noise or sampling, then the plant would indeed act as a perfect integral of order n regardless of $G_{\rm p}$. By rearranging (11) as (13), the transition from the desired integral form at low frequency to the original plant at the high frequency can be captured by the transfer function of $\bar{G}_{\rm p}(s)$ in the form of (13), where a low-pass filter shapes the plant into the integral form at low frequency and a high-pass filter shapes the plant at high frequency.

$$\bar{G}_{\rm p}(s) = \frac{1}{s^n\left[\dfrac{1}{(s/\omega_{\rm o}+1)^{n+1}}\right] + G_{\rm p}{}^{-1}(s)b_0\left[\dfrac{(s/\omega_{\rm o}+1)^{n+1}-1}{(s/\omega_{\rm o}+1)^{n+1}}\right]}. \tag{13}$$

As $\omega_{\rm o}$ is tuned, the poles and zeros, as well as the shape of the frequency response of $\bar{G}_{\rm p}(s)$, change with it. With $n = 1$, $G_{\rm p}(s) = \dfrac{b_0}{s+3}$, and the second order ESO in the form of (3), the pole movements of (13) are shown graphically in the root locus plot of Fig. 4, where the observer bandwidth is varied between 0 and whatever value that causes one of the poles to reach a radius of 10 from the origin.

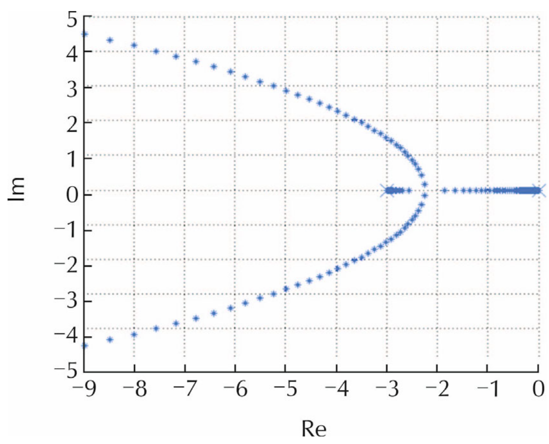


Fig. 4 Closed-loop poles for the 1st order plant as $\omega_{\rm o}$ varies.

This pattern is the same for any stable first-order plant. By monitoring how the poles of the modified plant move, it can be better understood how ADRC forces the plant to behave like cascaded integrators. The information about the imperfection can be used in the control design to better accommodate the remaining dynamics beyond the cascaded integrators.

4 Time domain and frequency domain connection

In [41], the ESO and the associated controller were parameterized, thusly LESO and LADRC were formulated. In that paper, the observer bandwidth and controller bandwidth, which engineers are familiar with, were first connected to ESO and ADRC as the tuning paramters. The parameterization of ESO and ADRC makes the concept very easy to understand and implement by engineers, therefore widely used in practice [46–48]. Critical to the connection between time domain formulation and frequency domain insights is established through the use of bandwidth in time domain analysis.

With the parameterized ADRC, the first attempt at the rigorous study of stability of the ADRC solution can be found in [34]. For the first time the convergence of the ESO and the bound on the tracking error in the ADRC were established by solving differential equations. The detailed derivations are given in [49].

4.1 Convergence of the ESO error dynamics

First, consider the ESO with the given model of the plant. Let $\tilde{x}_i = x_i - \hat{x}_i, i = 1, 2, \ldots, n+1$. From (2) and (6), the observer estimation error for the system with a given model can be shown as

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 - \omega_{\rm o}\alpha_1\tilde{x}_1, \\ \vdots \\ \dot{\tilde{x}}_{n-1} = \tilde{x}_n - \omega_{\rm o}^{n-1}\alpha_{n-1}\tilde{x}_1, \\ \dot{\tilde{x}}_n = \tilde{x}_{n+1} - \omega_{\rm o}^n\alpha_n\tilde{x}_1, \\ \dot{\tilde{x}}_{n+1} = h(x, u, w, \dot{w}) - h(\hat{x}, u, w, \dot{w}) - \omega_{\rm o}^{n+1}\alpha_{n+1}\tilde{x}_1. \end{cases} \tag{14}$$

Now let $\varepsilon_i = \dfrac{\tilde{x}_i}{\omega_{\rm o}^{i-1}}$, $i = 1, 2, \ldots, n+1$, then (14) can be rewritten as

$$\dot{\varepsilon} = \omega_{\rm o}A\varepsilon + B\frac{h(x, u, w, \dot{w}) - h(\hat{x}, u, w, \dot{w})}{\omega_{\rm o}^n}, \tag{15}$$

where $A = \begin{bmatrix} -\alpha_1 & 1 & 0 \cdots & 0 \\ -\alpha_2 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_n & 0 & 0 \cdots & 1 \\ -\alpha_{n+1} & 0 & 0 \cdots & 0 \end{bmatrix}$, $B = [0\ 0\ \cdots\ 0\ 1]^{\mathrm{T}}$. Here, A is Hurwitz for the $\alpha_i, i = 1, 2, \ldots, n+1$, chosen above.

Theorem 1 Assuming $h(x, u, w, \dot{w})$ is globally Lipschitz with respect to x, there exists a constant $\omega_{\mathrm{o}} > 0$, such that $\lim_{t\to\infty} \tilde{x}_i(t) = 0, i = 1, 2, \ldots, n+1$.

The proof of this theorem has been given in [50].

When the plant dynamics is largely unknown, the ESO is designed as shown in (3). Consequently, the observer estimation error becomes

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 - \omega_{\mathrm{o}}\alpha_1\tilde{x}_1, \\ \vdots \\ \dot{\tilde{x}}_{n-1} = \tilde{x}_n - \omega_{\mathrm{o}}^{n-1}\alpha_{n-1}\tilde{x}_1, \\ \dot{\tilde{x}}_n = \tilde{x}_{n+1} - \omega_{\mathrm{o}}^{n}\alpha_n\tilde{x}_1, \\ \dot{\tilde{x}}_{n+1} = h(x, u, w, \dot{w}) - \omega_{\mathrm{o}}^{n+1}\alpha_{n+1}\tilde{x}_1, \end{cases} \tag{16}$$

and equation (15) is now

$$\dot{\varepsilon} = \omega_{\mathrm{o}} A\varepsilon + B\frac{h(x, u, w, \dot{w})}{\omega_{\mathrm{o}}^n}. \tag{17}$$

Theorem 2 Assuming $h(x, u, w, \dot{w})$ is bounded, there exist $\omega_{\mathrm{o}} > 0$, a constant $\sigma_i > 0$, and a finite $T_1 > 0$ such that $|\tilde{x}_i(t)| \leqslant \sigma_i, i = 1, 2, \ldots, n+1, \forall t \geqslant T_1$. Furthermore, $\sigma_i = \mathrm{O}(\frac{1}{\omega_{\mathrm{o}}^k})$, for some positive integer k.

The proof of this theorem has been given in [49].

In summary, when the plant model is given and used in the ESO, the dynamic system describing the ESO estimation error is asymptotically stable; and in the absence of such model, the ESO estimation error is bounded and its upper bound monotonously decreases with the observer bandwidth. The stability characteristics of the ADRC, where the ESO is employed, is presented next.

4.2 Stability characteristics of the ADRC

In [49], the stability characteristics of the ADRC are presented for both the cases of the plant model given and plant dynamics largely unknow.

Define $e_i = r_i - x_i, i = 1, 2, \ldots, n$. In (7), $k_i, i = 1, 2, \ldots, n$, are selected such that the characteristic polynomial $s^n + k_n s^{n-1} + \ldots + k_1$ is Hurwitz. For tuning simplicity, we just let $s^n + k_n s^{n-1} + \ldots + k_1 = (s + \omega_{\mathrm{c}})^n$ where $\omega_{\mathrm{c}} > 0$ and $k_i = \dfrac{n!}{(i-1)!(n+1-i)!}\omega_{\mathrm{c}}^{n+1-i}$, $i = 1, 2, \ldots, n$. This makes ω_{c}, which is the controller bandwidth, the only tuning parameter to be adjusted for the controller.

For the case of the plant model given, one has the following theorem.

Theorem 3 Assuming $h(x, u, w, \dot{w})$ is globally Lipschitz with respect to x, there exist constants $\omega_{\mathrm{o}} > 0$ and $\omega_{\mathrm{c}} > 0$, such that the closed-loop system (8) is asymptotically stable.

Now we consider the case that the plant dynamics is unknown and the ESO in the form of (3) is used instead.

Theorem 4 Assuming $h(x, u, w, \dot{w})$ is bounded, there exist $\omega_{\mathrm{c}} > 0$, $\omega_{\mathrm{o}} > 0$, a constant $\rho_i > 0$, and a finite time $T_5 > 0$ such that $|e_i(t)| \leqslant \rho_i, i = 1, 2, \ldots, n, \forall t \geqslant T_5$. Furthermore, $\rho_i = \mathrm{O}(\frac{1}{\omega_{\mathrm{c}}^j})$ for some positive integer j.

The proof of Theorems 3 and 4 can be found in [49]. In summary, with the given model of the plant, the closed-loop system (8) is asymptotically stable; and with plant dynamics largely unknown, the tracking error and its up to $(n-1)$th order derivatives of the ADRC are bounded and their upper bounds monotonously decrease with the observer and controller bandwidths.

The above analyses show that the observer bandwidth and control loop bandwidth are associated in the time domain stability analysis with the upper bounds of the observer error and the tracking error, respectively. This makes such analysis relevant to the common design considerations and concerns shared by practicing engineers.

5 Time domain validation

With the convergence of the ESO and the ADRC established, a simulation study of nonlinear plant with partial model information is presented below.

Consider the following nonlinear system

$$\ddot{y} = \dot{y}^3 + y + d + u. \tag{18}$$

Rewrite (18) as

$$\ddot{y} = f + bu, \tag{19}$$

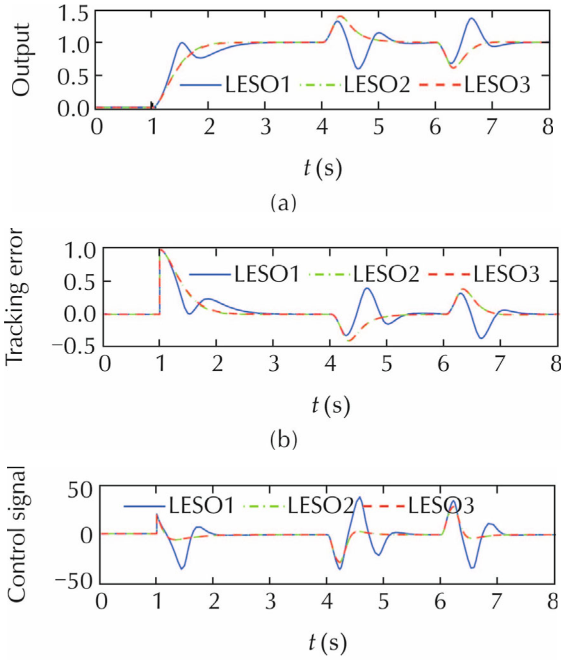
where f represents the summation of the plant dynamics $\dot{y}^3 + y$ and the external disturbance d.

Note that for a second order plant, the LESO in (6) and (3) is of the third order, where $\hat{x}_3$ is an estimate of f. With a well-tuned observer, the control law is given by

$$u = \frac{k_1(r - \hat{x}_1) + k_2(\dot{r} - \hat{x}_2) - \hat{x}_3 + \ddot{r}}{b}, \tag{20}$$

where $k_1 = \omega_c^2$, and $k_2 = 2\omega_c$.

The LADRC tracking performance is shown in Fig. 5 under three different scenarios: 1) f is completely unknown; 2) only partial internal dynamics information of the plant is given, i.e., $f_{\text{partial}} = \dot{y}^3$; 3) the internal dynamics of the plant f_{in} is completely known, i.e., $f_{\text{in}} = \dot{y}^3 + y$ is given. In this simulation, the tuning parameters are $\omega_c = 4.5$ rad/s and $\omega_o = 20$ rad/s. Fig. 5 shows the tracking errors between the reference and the output for three cases using a step input at $t = 1$ s as the excitation and a pulse disturbance with the amplitude of ±20, the period of 4 s, the pulse width 5% of the period, and the phase delay of 4 s. From Fig. 5, it can be observed that the tracking error of the control loop decreases as more model information is incorporated into the LADRC.



(c)

Fig. 5 The LADRC performance with different LESOs for the nonlinear system (LESO1: without plant information; LESO2: with partial plant information, i.e., $f_{\text{partial}} = \dot{y}^3$; LESO3: with complete plant information, i.e., $f_{\text{in}} = \dot{y}^3 + y$ is given).

Note that the system (18) is a nonlinear system. The above simulation demonstrates that ADRC can control the nonlinear system with large uncertainties very well although the ADRC itself is linear. Many other approaches can deal with uncertainties, however, most of them can only handle small uncertainties. ADRC is a very simple and straightforward approach. It is easy to understand by engineers and easy to implement in real applications. From the above simulation, even $\dot{f}$, i.e., h tends to ∞, the ESO gain is still low. Therefore, the ESO is not a high gain observer.

6 Conclusions

In this paper, the time domain formulation and frequency domain understanding of the ADRC are connected. It is shown that the formulation of the ADRC in time domain can be easily understood by engineers with insights in the language of frequency responses, such as the bandwidth and stability margins. From both the frequency responses and the time domain validation, it is clear that the ADRC is unique in its ability of disturbance rejection and in its robustness to large uncertainties in process dynamics. It also shows that the stability characteistics of the ESO and the ADRC can be analyzed directly by solving the differential equations, instead of indirectly by using the standard techniques such as the Lyapunov methods. In doing so, the relationship between the error bounds and the ADRC bandwidth is disclosed. In the ADRC analysis and validation, one can see that the ADRC can handle nonlinear systems with large uncertainties and disturbances without the need of accurate mathematical model of the plant. Partial model information, if given, can and should be incoporated into the ESO for better performance, less noise sensitivity and the reduced bandwidth.

References

[1] R. Brockett. New issues in the mathematics of control. *Mathematics Unlimited – 2001 and Beyond*. B. Engquist, W. Schimid (eds). Berlin: Springer, 2001: 189 – 220.

[2] J. Han. Is it control theory or model theory? *Systems Science and Mathematical Sciences*, 1989, 9(4): 328 – 335 (in Chinese).

[3] J. Han. A class of extended state observers for uncertain systems. *Control and Decision*, 1995, 10(1): 85 – 88 (in Chinese).

[4] J. Han. Nonlinear state error feedback control. *Control and Decision*, 1995, 10(3): 221 – 225 (in Chinese).

[5] J. Han. Auto-disturbance rejection control and its applicationss. *Control and Decision*, 1998, 13(1): 19 – 23 (in Chinese).

[6] J. Han. Nonlinear design methods for control systems. *Proceedings of the 14th IFAC World Congress*, Beijing, 1999: 521 – 526.

[7] Z. Gao, Y. Huang, J. Han. An alternative paradigm for control system design. *Proceedings of the IEEE Conference on Decision and Control*, New York: IEEE, 2001: 4578 – 4585.

[8] Z. Gao. Active disturbance rejection control: a paradigm shift

in feedback control system design. *Proceedings of the American Control Conference*, New York: IEEE, 2006: 2399 – 2405.

[9] Z. Gao. On the centrality of disturbance rejection in automatic control. *ISA Transactions*, 2014, 53(4): 850 – 857.

[10] Z. Gao. Active disturbance rejection control: from an enduring idea to an emerging technology. *Proceedings of the 10th International Workshop on Robot Motion and Control*, Poznan: IEEE, 2015: 269 – 282.

[11] J. Han. From PID to active disturbance rejection control. *IEEE Transactions on Industrial Electronics*, 2009, 56(3): 900 – 906.

[12] G. Basile, G. Marro. On the observability of linear, time-invariant systems with unknown inputs. *Journal of Optimization Theory and Applications*, 1969, 2(6): 410 – 415.

[13] C. Johnson. Accommodation of external disturbances in linear regulator and servomechanism problems. *IEEE Transactions on Automatic Control*, 1971, 16(6): 635 – 644.

[14] G. Hostetter, J. Meditch. On the generalization of observers to systems with unmeasurable, unknown inputs. *Automatica*, 1973, 9(6): 721 – 724.

[15] C. Johnson. Theory of disturbance-accommodating controllers. *Control and Dynamic Systems*, 1976, 12: 387 – 489.

[16] V. Gourishangkar, P. Kudva, K. Ramar. Reduced-order observers for multivariable systems with inaccessible disturbance inputs. *International Journal of Control*, 1977, 25(2): 311 – 319.

[17] P. C. Muller. Indirect measurements of nonlinear effects by state observers. *Nonlinear Dynamics in Engineering Systems*, Berlin: Springer, 1990: 205 – 215.

[18] J. Profeta, W. Vogt, M. Mickle. Disturbance estimation and compensation in linear systems. *IEEE Transaction on Aerospace and Electronic Systems*, 1990, 26(2): 225 – 231.

[19] J. Chen, R. J. Patton, H. Zhang. Design of unknown input observers and robust fault detection filters. *International Journal of Control*, 1995, 63(1): 85 – 105.

[20] T. Umeno, Y. Hori. Robust speed control of DC servo motors using modern two degrees-of-freedom controller design. *IEEE Transactions on Industrial Electronics*, 1991, 38(5): 363 – 368.

[21] Y. Hori, K. Shimura, M. Tomizuka. Position/force control of multi-axis manipulator based on the TDOF robust servo controller for each joint. *Proceedings of the American Control Conference ACC/WM9*, New York: IEEE, 1992: 753 – 757.

[22] H. Lee, M. Tomizuka. Robust motion controller design for high-accuracy positioning systems. *IEEE Transactions on Industrial Electronics*, 1996, 43(1): 48 – 55.

[23] T. Mita, M. Hirata, K. Murata, et al. H_∞ control versus disturbance-observer-based control. *IEEE Transactions on Industrial Electronics*, 1998, 45(3): 488 – 495.

[24] R. Bickel, M. Tomizuka. Passivity-based versus disturbance observer based robot control: equivalence and stability. *ASME Journal of Dynamics Systems, Measurement, and Control*, 1999, 121(1): 41 – 47.

[25] E. Schrijver, J. van Dijk. Disturbance observers for rigid mechanical systems: equivalence, stability, and design. *ASME Journal of Dynamics Systems, Measurement, and Control*, 2002, 124(4): 539 – 548.

[26] Y. Choi, K. Yang, W. K. Chung, et al. On the robustness and performance of disturbance observers for second-order systems. *IEEE Transactions on Automatic Control*, 2003, 48(2): 315 – 320.

[27] K. Yang, Y. Choi, W. Chung. On the tracking performance improvement of optical disk drive servo systems using error-based disturbance observer. *IEEE Transactions on Industrial Electronics*, 2005, 52(1): 270 – 279.

[28] S. Kwon, W. K. Chung. Robust performance of the multiloop perturbation compensator. *IEEE/ASME Transaction on Mechatronics*, 2002, 7(2): 190 – 200.

[29] S. Kwon, W. K. Chung. A discrete-time design and analysis of perturbation observer for motion control applications. *IEEE Transactions on Control Systems Technology*, 2003, 11(3): 399 – 407.

[30] S. Kwon, W. K. Chung. Combined synthesis of state estimator and perturbation observer. *ASME Journal of Dynamic Systems, Measurement, and Control*, 2003, 125(1): 19 – 26.

[31] S. Kwon. Robust Kalman filtering with perturbation estimation process. *Proceedings of the American Control Conference*, New York: IEEE, 2006: 997 – 1002.

[32] A. Radke, Z. Gao. A survey of state and disturbance observers for practitioners. *Proceedings of the American Control Conference*, New York: IEEE, 2006: 5183 – 5188.

[33] W. Chen, J. Yang, L. Guo, et al. Disturbance observer-based control and related methods: an overview. *IEEE Transactions on Industrial Electronics*, 2016, 63(2): 1083 – 1095.

[34] Q. Zheng, L. Gao, Z. Gao. On stability analysis of active disturbance rejection control for nonlinear time-varying plants with unknown dynamics. *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans: IEEE, 2007: 3501 – 3506.

[35] Z. Zhao, B. Guo. On convergence of nonlinear active disturbance rejection control for SISO nonlinear systems. *Journal of Dynamical and Control Systems*, 2016, 22(2): 385 – 412.

[36] Y. Huang, W. Xue. Active disturbance rejection control: methodology and theoretical analysis. *ISA Transactions*, 2014, 53(4): 963 – 976.

[37] B. Guo, H. Zhou. The active disturbance rejection control to stabilization for multi-dimensional wave equation with boundary control matched disturbance. *IEEE Transactions on Automatic Control*, 2015, 60(1): 143 – 157.

[38] B. Guo, F. Jin. The active disturbance rejection and sliding mode control approach to the stabilization of the Euler-Bernoulli beam equation with boundary input disturbance. *Automatica*, 2013, 49(9): 2911 – 2918.

[39] B. Guo, Z. Zhao. On convergence of the nonlinear active disturbance rejection control for MIMO systems. *SIAM Journal on Control and Optimization*, 2013, 51(2): 1727 – 1757.

[40] B. Guo, Z. Zhao. On the convergence of extended state observer for nonlinear systems with uncertainty. *Systems & Control Letters*, 2011, 60(6): 420 – 430.

[41] Z. Gao. Scaling and parameterization based controller tuning. *Proceedings of the American Control Conference*, New York: IEEE, 2003: 4989 – 4996.

[42] G. Tian, Z Gao. Frequency response analysis of active disturbance rejection based control system. *IEEE Multi-conference on Systems and Control*, New York: IEEE, 2007: 1167 – 1172.

[43] J. Csank, Z Gao. Uncertainty reduction through active disturbance rejection. *Proceedings of the American Control Conference*, New York: IEEE, 2008: 3689 – 3694.

[44] J. Tatsumi, Z Gao. On the enhanced ADRC design with a low observer bandwidth. *Proceedings of the Chinese Control Conference*, New York: IEEE, 2013: 297 – 302.

[45] W. Xue, Y. Huang. Performance analysis of active disturbance rejection tracking control for a class of uncertain LTI systems. *ISA Transactions*, 2015, 58: 133 – 154.

[46] Q. Zheng, L. Dong, D. H. Lee, et al. Active disturbance rejection control and implementation for MEMS gyroscopes. *IEEE Transactions on Control Systems Technology*, 2009, 17(6): 1432 – 1438.

[47] Q. Zheng, Z. Chen, Z Gao. A practical dynamic decoupling control approach. *Control Engineering Practice*, 2009, 17(9): 1016 – 1025.

[48] Q. Zheng, Z. Gao. An energy saving, factory-validated disturbance decoupling control design for extrusion processes. *Proceedings of the 10th World Congress on Intelligent Control and Automatio*, Beijing: IEEE, 2012: 2891 – 2896.

[49] Q. Zheng. *On Active Disturbance Rejection Control: Stability Analysis and Applications in Disturbance Decoupling Control*. Ph.D. thesis. Cleveland: Cleveland State University, 2009.

[50] Q. Zheng, L. Gao, Z. Gao. On validation of extended state observer through analysis and experimentation. *ASME Journal of Dynamic Systems, Measurement, and Control*, 2012, 134(2): DOI 10.1115/1.4005364.

Qing ZHENG is an associate professor in the Electrical and Computer Engineering Department at Gannon University, received the Doctor of Engineering degree in Electrical Engineering from Cleveland State University in 2009. Before joining Gannon University, she had been employed at Beijing Institute of Control Devices in China for five years. Her research interests include active disturbance rejection control and its applications as well as multivariable control and optimization. E-mail: zheng003@gannon.edu.

Zhiqiang GAO received the Ph.D. degree in Electrical Engineering from University of Notre Dame in 1990. He is an associate professor and the director of Center for Advanced Control Technologies, Cleveland State University, His research is focused on the principles and practice of engineering cybernetics, particularly its manifestation in active disturbance rejection control. E-mail: z.gao@ieee.org.