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A new semi-tensor product of matrices

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Abstract

A new matrix product, called the second semi-tensor product (STP-II) of matrices is proposed. It is similar to the classical semi-tensor product (STP-I). First, its fundamental properties are presented. Then, the equivalence relation caused by STP-II is obtained. Using this equivalence, a quotient space is also obtained. Finally, the vector space structure, the metric and the metric topology, the projection and subspaces, etc. of the quotient space are investigated in detail.

Keywords: Second sime-tensor product (STP-II), equivalence class, quotient space, topology, metric

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1 Introduction

As a generalization of conventional matrix product, the semi-tensor product (STP-I) is defined as follows:

Definition 1 Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $t = n \vee p$ be the least common multiple of n and p. Then the left semi-tensor product (STP-I) of A and B, denoted by $A \ltimes B$, is defined as

$$A \ltimes B := (A \otimes I_{t/n})(B \otimes I_{t/p}). \tag{1}$$

The right STP-I of A and B, denoted by $A \rtimes B$, is defined as

$$A \rtimes B := (I_{t/n} \otimes A)(I_{t/p} \otimes B). \tag{2}$$

We use $\bowtie$ for either $\ltimes$ or $\rtimes$. Since $\ltimes$ has better properties than $\rtimes$, STP-I is defaulted to be left STP-I next unless elsewhere stated.

When $n = p$ STP-I becomes the conventional matrix product. Hence it is a generalization of conventional matrix product. Though the conventional matrix product has been extended to STP-I, which is applicable to two arbitrary matrices, its all properties remain available.

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STP-I was firstly proposed in 2001 [1]. Since then it has received many applications, and becomes an important tool in stabilization and control design of dynamic systems, e.g., power systems [2]; analysis and control of logical systems [3]; finite games [4, 5]; etc.

In recent study on cross-dimensional linear systems [6], a projection $\mathrm{Pr} : \mathcal{M}_{m\times m} \to \mathcal{M}_{km\times km}$ is proposed as

$$\mathrm{Pr}(A) := A \otimes J_k, \tag{3}$$

where

$$J_k := \frac{1}{k}\mathbf{1}_{k\times k}, \tag{4}$$

that is, J_k is a $k \times k$ matrix with $1/k$ as its all entries. This fact motivates us to define the STP-II as follows.

Definition 2 Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $t = n \vee p$ be the least common multiple of n and p. Then the second left (right) semi-tensor product (STP-II) of A and B, denoted by $A \circ_\ell B$ (correspondingly, $A \circ_r B$), is defined as

$$\begin{cases} A \circ_\ell B := (A \otimes J_{t/n})(B \otimes J_{t/p}), \\ A \circ_r B := (J_{t/n} \otimes A)(J_{t/p} \otimes B). \end{cases} \tag{5}$$

We use $\circ$ for both left and right STP-II. It is obvious that $\circ$ is also a generalization of conventional matrix product.

We may also call STP-I and STP-II by first matrix-matrix (MM-1) product and second matrix-matrix (MM-2) product, because both of them are matrix-matrix product. The purpose of this paper is to introduce the concept of STP-II, and then to investigate the properties of STP-II. Some related topics are also discussed, which include 1) equivalence and quotient space; 2) lattice structure of equivalence class; 3) vector space structures of matrices and their equivalence classes; 4) matric space structure and its matric topology; and 5) projection and subspaces. Most of the results are presented without proofs, because they have their corresponding known results for first STP. Then the original proofs can easily be revised to provide proofs for corresponding results of STP-II.

It seems to us that STP-II is also a very useful tool in investigating cross dimensional dynamic (control) systems [6].

The rest of this paper is organized as follows: Section 2 provides some fundamental properties of STP-II. Section 3 proposes an equivalence called the STP-II equivalence. Based on the STP-II equivalence, a quotient space is obtained in Section 4. Section 5 proposes a vector space structure on quotient space. Sections 6 and 7 consider the metrics on matrix space and its quotient space respectively. The metric topology is also presented. Section 8 investigates the subspaces of quotient space. Section 9 is a brief conclusion.

2 Some fundamental properties

This section considers some basic properties of STP-II. Most of them have their corresponding ones for STP-I.

First, we consider J_k. Some of its properties, mentioned in the follows, are easily verifiable. Hence, the proofs are omitted.

Proposition 1

1)

$$J_p \otimes J_q = J_{pq}. \tag{6}$$

2)

$$J_p \cdot J_p = J_p.$$

3)

$$\mathrm{rank}(J_k) = 1, \quad \forall k.$$

4) J_k has only one non-zero eigenvalue, which is 1.

5) $\{J_k \mid k \in \mathbb{N}\}$ is a monoid with identity $J_1 = 1$.

Note that a set G with a binary operation $* : G \times G \to G$, denoted by $(G, *)$, is a monoid, if it is a semigroup with identity [7].

We list some properties of STP-II without proof. Because they are similar to the corresponding ones of STP, only by replacing I_k by J_k. Then their proofs are almost the same as for STP except some obvious modifications.

Proposition 2 Next, $\circ$ can be either $\circ_\ell$ or $\circ_r$.

1) (Associativity)

$$(A \circ B) \circ C = A \circ (B \circ C). \tag{7}$$

2) (Distributivity)

$$\begin{cases} A \circ (\alpha B + \beta C) = \alpha A \circ B + \beta A \circ C, \\ (\alpha A + \beta B) \circ C = \alpha A \circ C + \beta B \circ C. \end{cases} \tag{8}$$

3)

$$(A \circ B)^{\mathrm{T}} = B^{\mathrm{T}} \circ A^{\mathrm{T}}. \tag{9}$$

Proposition 3 Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Then the left STP-II of A and B can be alternatively defined as

$$A \circ_\ell B = (C_{i,j}), \tag{10}$$

where

$$C_{ij} = \mathrm{Row}_i(A) \circ_\ell \mathrm{Col}_j(B), \quad i = 1, \dots, m; \ j = 1, \dots, q.$$

However, this is not true for $\circ_r$.

Remark 1 Denote the set of all matrices as

$$\mathcal{M} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m\times n}. \tag{11}$$

It is easy to verify that $(\mathcal{M}, \circ)$ is a semi-group. However, unlike $(\mathcal{M}, \bowtie)$, $(\mathcal{M}, \circ)$ is not a monoid, because $1 \circ A \neq A$. The scalar product and $\bowtie$ are consistent, but the number product and $\circ$ are not consistent. This is a big difference between $(\mathcal{M}, \circ)$ and $(\mathcal{M}, \bowtie)$. (Note that similar to $\circ$, $\bowtie$ can be either $\ltimes$ or $\rtimes$).

3 STP-II equivalence

Similarly to STP-I, one sees easily that STP-II is basically also a product of two equivalence classes $\{A, A \circ J_1, A \circ J_2, \dots\}$ with $\{B, B \circ J_1, B \circ J_2, \dots\}$. Motivated by this, we give the following equivalence relation.

Definition 3

- $A, B \in \mathcal{M}$ are said to be left STP-II equivalent, denoted by $A \approx_\ell B$, if there exist J_α and J_β, such that

$$A \otimes J_\alpha = B \otimes J_\beta. \tag{12}$$

- $A, B \in \mathcal{M}$ are said to be right STP-II equivalent, denoted by $A \approx_r B$, if there exist J_α and J_β, such that

$$J_\alpha \otimes A = J_\beta \otimes B. \tag{13}$$

- The left STP-II equivalent class is denoted by

$$\hat{A}_\ell := \{B \,|\, B \approx_\ell A\}.$$

- The right STP-II equivalent class is denoted by

$$\hat{A}_r := \{B \,|\, B \approx_r A\}.$$

Denote by

$$\Xi_\ell = \mathcal{M}/\approx_\ell; \quad \Xi_r = \mathcal{M}/\approx_r,$$

which are the set of left, right STP-II equivalence classes respectively.

It is ready to verify that the relations defined by (12) and (13) are equivalence relations, (i.e., both of them are reflexive, symmetric, and transitive [8]).

Let A be a square matrix. Then it is ready to check that

$$\mathrm{tr}(A \otimes J_k) = \mathrm{tr}(J_k \otimes A) = \mathrm{tr}(A). \tag{14}$$

Hence, we can define the trace on an equivalence class as follows.

Definition 4 Consider an equivalence class of square matrix $\hat{A}$, which is either $\hat{A}_\ell$ or $\hat{A}_r$. A trace is defined by

$$\mathrm{tr}(\hat{A}) := \mathrm{tr}(A). \tag{15}$$

Note that $\hat{A}$ here can be either $\hat{A}_\ell$ or $\hat{A}_r$.

4 Lattice structure on equivalence class

For statement and notational ease, hereafter $\circ$ is understood as $\circ_\ell$ and $\approx$ is understood as $\approx_\ell$. Then we do not need to repeat similar statements twice. With obvious modification one sees easily that $\circ$ could be understood as for both $\circ_\ell$ and $\circ_r$.

Definition 5 [9] Let L be a partial ordered set. If for any two elements $a, b \in L$ there exist a lowest upper bound $\sup(a, b) \in L$ and a greatest lower bound $\inf(a, b) \in L$, then L (with the order) is called a lattice.

Definition 6 Assume $A \approx B$ and there exists J_k, $k \geqslant 1$, such that $A \otimes J_k = B$, then

1) A is called a divisor of B and B is called a multiple of A;

2) an order can be defined as $A \prec B$. This order makes $\mathcal{M}$ a partial order set.

Proposition 4 Assume $A \approx B$, and hence (12) holds. If in (12) $\alpha \wedge \beta = 1$. Define

$$\Theta = A \otimes J_\alpha = B \otimes J_\beta. \tag{16}$$

Then $\Theta = \sup(A, B)$. That is, Θ is the least common multiple of A and B.

Proposition 5 Assume $A \approx B$. Then there exists a Λ, such that

$$A = \Lambda \otimes J_a, \quad B = \Lambda \otimes J_b. \tag{17}$$

Assume $a \wedge b = 1$, then $\Lambda = \inf(A, B)$. That is, Λ is the greatest common divisor of A and B.

Propositions 4 and 5 assure the following lattice structure.

Corollary 1 Let $\bar{A} \in \Xi$. Then $(\bar{A}, \prec)$ is a lattice.

In previous Corollary 1 $\bar{A}$ could be either $\bar{A}_\iota$ or $\bar{A}_{\rm r}$. Then Ξ is either Ξ_ι or $\Xi_{\rm r}$. Hence, we have two kinds of lattices: $(\bar{A}_\iota, \prec_\iota)$ and $(\bar{A}_{\rm r}, \prec_{\rm r})$.

Remark 2 The following results about STP-I equivalence are well known [10].

1) $A, B \in \mathcal{M}$ is said to be STP equivalent, denoted by $A \sim B$, if there exist I_α and I_β such that

$$A \otimes I_\alpha = B \otimes I_\beta. \tag{18}$$

The equivalence class is denoted by

$$\langle A \rangle := \{B \,|\, B \sim A\}.$$

2) A partial order $\prec$ of $\mathcal{M}$ is defined as follows: $A \prec B$, if there exists I_k such that $A \otimes I_k = B$.

3) Assume (18) holds and $\alpha \wedge \beta = 1$, set

$$\Theta := A \otimes I_\alpha = B \otimes I_\beta.$$

Then $\Theta = \sup(A, B)$.

4) Assume $A \sim B$, then there exists a $\Lambda \in \mathcal{M}$, such that

$$A = \Lambda \otimes I_p, \;\; B = \Lambda \otimes I_q.$$

If $p \wedge q = 1$, then Λ is unique. Moreover, $\Lambda = \inf(A, B)$.

We summarize that $(\langle A \rangle, \prec)$ is a lattice.

All the above statements have been proved in [10]. As the I_k's being replaced by J_k's, all the proofs remain available. Therefore, we skip the proofs for Propositions 4, 5, and Corollary 1.

Proposition 6 Let $(\langle A \rangle, \prec)$ and $(\bar{B}, \prec)$ be two lattices. Then $(\langle A \rangle, \prec)$ and $(\bar{B}, \prec)$ are isomorphic lattices.

Proof Let $A_1 \in \langle A \rangle$ and $B_1 \in \bar{B}$ be the root elements of $\langle A \rangle$ and $\bar{B}$ respectively. Define $\varphi : \langle A \rangle \to \bar{B}$ as follows:

$$\varphi(A_1 \otimes I_k) := B_1 \otimes J_k, \;\; k = 1, 2, \ldots.$$

It is ready to verify that φ is a lattice isomorphism. □

5 Vector space structure

Define

$$\mathcal{M}_\mu := \{A \in \mathcal{M}_{m\times n} \,|\, m/n = \mu\}.$$

Then we have a partition as

$$\mathcal{M} = \bigcup_{\mu \in \mathbb{Q}_+} \mathcal{M}_\mu, \tag{19}$$

where $\mathbb{Q}_+$ is the set of positive rational numbers. Correspondingly, we also set

$$\Xi_\mu := \mathcal{M}_\mu / \approx .$$

Then we also have a partition for the quotient space as

$$\Xi = \bigcup_{\mu \in \mathbb{Q}_+} \Xi_\mu. \tag{20}$$

Our purpose is to pose a proper vector space structure on each $\mathcal{M}_\mu$ and Ξ_μ.

Definition 7 [11] Let X be a set . Suppose there is a mapping $(x, y) \mapsto x + y$ of $X \times X$ into X, called addition, and a mapping $(a, x) \mapsto ax$ of $\mathbb{R} \times X$ into X, called scalar multiplication, such that the following axioms are satisfied (x, y, z denoting arbitrary elements of L, and a, b arbitrary elements of $\mathbb{R}$):

1) $(x + y) + z = z + (y + z)$,

2) $x + y = y + x$,

3) There exists a unique element $0 \in X$, such that $x + 0 = x$ for all $x \in X$,

4) For each $x \in X$, there exists unique $z = -x \in X$ such that $x + z = 0$,

5) $a(x + y) = ax + ay$,

6) $(a + b)x = ax + bx$,

7) $a(bx) = (ab)x$,

8) $1x = x$,

then X is called a vector space.

Definition 8 Assume X with addition "+" and scalar multiplication "·" satisfies all the requires for a vector space except that the zero is a set, hence for each $x \in X$ the inverse $-x$ may not unique. Then X is called a pseudo-vector space.

Definition 9 Let $A, B \in \mathcal{M}_\mu$. Precisely, $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $m/n = p/q = \mu$. Set $t = m \vee p$. Then,

1) the left M-II matrix addition of A and B, denote by $+_\iota$, is defined as

$$A \;+_\iota\; B := (A \otimes J_{t/m}) + (B \otimes J_{t/p}). \tag{21}$$

Correspondingly, the left M-II matrix subtraction is defined as

$$A -_\ell B := A +_\ell (-B); \tag{22}$$

2) the right M-II matrix addition of A and B, denote by $+_r$, is defined as

$$A +_r B := (J_{t/m} \otimes A) + (J_{t/p} \otimes B). \tag{23}$$

Correspondingly, the right M-II matrix subtraction is defined as

$$A -_r B := A +_r (-B). \tag{24}$$

Remark 3 If in the above definition all J_k's are replaced by corresponding I_k's, the M-II matrix addition/subtraction becomes M-I matrix addition/subtraction, which have been discussed in detail in [10].

Remark 4 Let $\sigma \in \{+_\ell, -_\ell, +_r, -_r\}$ be one of the four binary operators. Then it is easy to verify that

1) if $A, B \in \mathcal{M}_\mu$, then $A\sigma B \in \mathcal{M}_\mu$;

2) if A and B are as in Definition 9, then $A\sigma B \in \mathcal{M}_{t\times \frac{t}{\mu}}$;

3) set $s = n \vee q$, then $s/n = t/m$ and $s/q = t/p$. Therefore, σ can also be defined by using column numbers respectively, e.g.,

$$A +_\ell B := (A \otimes I_{s/n}) + (B \otimes I_{s/q}),$$

etc.

It is easy to verify the following conclusion.

Proposition 7 $\mathcal{M}_\mu$ with addition ($+_\ell$ or $+_r$) and conventional scalar product is a pseudo-vector space, where for each A, its inverse is defined as

$$-A := \{B \mid A +_\ell B = 0\}, \tag{25}$$

which is not unique.

In fact, it is easy to verify that $A +_\ell B = 0$, if and only if, $A \approx_\ell B$, (or $A +_r B = 0$, if and only if, $A \approx_r B$). Then when the quotient space is considered, we have a vector space.

Definition 10 Let $\bar{A}, \bar{B} \in \Xi$. Then,

$$\bar{A} +_\ell \bar{B} := \overline{A +_\ell B}, \tag{26}$$

$$\bar{A} +_r \bar{B} := \overline{A +_r B}. \tag{27}$$

Correspondingly,

$$\bar{A} -_\ell \bar{B} := \overline{A -_\ell B}, \tag{28}$$

$$\bar{A} -_r \bar{B} := \overline{A -_r B}. \tag{29}$$

It is easy to verify that (26) (or (27)) and (28) (or (29)) are properly defined. That is, they are independent of the choice of representatives $A \in \bar{A}$ and $B \in \bar{B}$. Moreover, the scalar product can be properly defined by

$$c\bar{A} = \overline{cA}, \quad c \in \mathbb{R}. \tag{30}$$

Finally, we have the following result:

Proposition 8 Ξ_μ with addition defined by (26) (or (27)) and scalar product defined by (30) is a vector space.

6 Metric and metric topology

Let $A = (a_{i,j}),\ B = (b_{i,j}) \in \mathcal{M}_{m\times n}$. It is well known that the Frobenius inner product of A and B is defined by

$$(A \mid B)_F = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}b_{i,j}. \tag{31}$$

The Frobenius norm is defined by

$$\|A\|_F = \sqrt{(A \mid A)_F}. \tag{32}$$

Now inside each $\mathcal{M}_\mu$ we assume $\mu_y, \mu_x \in \mathbb{N}$ are co-prime and $\mu_y/\mu_x = \mu$. Then,

$$\mathcal{M}_\mu = \bigcup_{i=1}^{\infty} \mathcal{M}_\mu^i,$$

where

$$\mathcal{M}_\mu^i = \mathcal{M}_{i\mu_y \times i\mu_x}, \quad i = 1, 2, \ldots.$$

Definition 11 Let $A, B \in \mathcal{M}_\mu$, where $A \in \mathcal{M}_\mu^\alpha$ and $B \in \mathcal{M}_\mu^\beta$. Then,

1) the left STP-II inner product of A, B is defined by

$$(A \mid B)_\ell := (A \otimes J_{t/\alpha} \mid B \otimes J_{t/\beta})_F, \tag{33}$$

where $t = \alpha \vee \beta$ is the least common multiple of α and β;

2) the left STP-II norm of A is defined by

$$\|A\|_\ell := \sqrt{(A \mid A)_\ell}; \tag{34}$$

3) a left STP-II matric (distance) of A and B is defined by

$$d_\iota(A, B) := \|A -_\iota B\|_\iota. \tag{35}$$

The corresponding right STP-II inner product, right STP-II norm, and right STP-II matric (distance) can be defined similarly.

The following proposition is easily verifiable.

Proposition 9 $\mathcal{M}_\mu$ with distance defined by (35) is a pseudo-metric space.

Remark 5 $\mathcal{M}_\mu$ with distance defined by (35) is not a metric space. It is easy to verify that $d_\iota(A, B) = 0$, if and only if, $A \approx_\iota B$.

Next, we consider the quotient space. We need the following lemma, which comes from a straightforward computation.

Lemma 1 Let $A, B \in \mathcal{M}_{m\times n}$. Then

$$(A \otimes J_k \,|\, B \otimes J_k)_{\rm F} = (A\,|\,B)_{\rm F}. \tag{36}$$

Using Lemma 1 and Definition 11, we have the following property.

Proposition 10 Let $A, B \in \mathcal{M}_\mu$, if A and B are orthogonal, i.e., $(A\,|\,B)_{\rm F} = 0$, then $A \otimes J_\xi$ and $B \otimes j_\xi$ are also orthogonal.

Now we are ready to define the inner product on Ξ_μ.

Definition 12 Let $\bar{A}, \bar{B} \in \Xi_\mu$. Their inner product is defined as

$$(\bar{A}\,|\,\bar{B})_\iota := (A\,|\,B)_\iota. \tag{37}$$

The following proposition shows that (37) is well defined.

Proposition 11 Definition 12 is well defined. That is, (37) is independent of the choice of representatives A and B.

Proof Assume $A_1 \in \bar{A}$ and $B_1 \in \bar{B}$ are irreducible. Then it is enough to prove that

$$(A\,|\,B)_\iota = (A_1\,|\,B_1)_\iota, \ \ A \in \bar{A},\ B \in \bar{B}. \tag{38}$$

Assume $A_1 \in \mathcal{M}_\mu^\alpha$ and $B_1 \in \mathcal{M}_\mu^\beta$. Let

$$\begin{aligned} A &= A_1 \otimes J_s \in \mathcal{M}_\mu^{\alpha s},\\ B &= B_1 \otimes J_t \in \mathcal{M}_\mu^{\beta t}. \end{aligned}$$

Denote by $\xi = \alpha \vee \beta$, $\xi\eta = \alpha s \vee \beta t$. Using (36), we have

$$\begin{aligned} (A\,|\,B)_\iota &= (A \otimes J_{\frac{\xi\eta}{\alpha s}} \,|\, B \otimes J_{\frac{\xi\eta}{\beta t}})_{\rm F}\\ &= (A_1 \otimes J_{\frac{\xi\eta}{\alpha}} \,|\, B_1 \otimes J_{\frac{\xi\eta}{\beta}})_{\rm F}\\ &= (A_1 \otimes J_{\frac{\xi}{\alpha}} \otimes J_\eta \,|\, B_1 \otimes J_{\frac{\xi}{\beta}} \otimes J_\eta)_{\rm F}\\ &= (A_1 \otimes J_{\frac{\xi}{\alpha}} \,|\, B_1 \otimes J_{\frac{\xi}{\beta}})_{\rm F}\\ &= (A_1\,|\,B_1)_\iota. \end{aligned}$$

□

Definition 13 [12] A real vector space X is an inner-product space, if there is a mapping $X\times X \to \mathbb{R}$, denoted by $(x|\ y)$, satisfying

1)

$$(x + y\,|\,z) = (x\,|\,z) + (y\,|\,z),\ \ x, y, z \in X.$$

2)

$$(x\,|\,y) = (y\,|\,x).$$

3)

$$(ax\,|\,y) = a(x\,|\,y),\ \ a \in \mathbb{R} \text{ (or } \mathbb{C}).$$

4)

$$(x\,|\,x) \geqslant 0, \text{ and } (x\,|\,x) \neq 0 \text{ if } x \neq 0.$$

By definition it is easy to verify the following result.

Theorem 1 The vector space $(\Xi_\mu, +_\iota)$ with the inner product defined by (37) is an inner product space.

Then the norm of $\bar{A} \in \Xi_\mu$ is defined naturally as

$$\|\bar{A}\|_\iota := \sqrt{(\bar{A}\,|\,\bar{A})_\iota}. \tag{39}$$

The following is some standard results for inner product space.

Theorem 2 Assume $\bar{A}, \bar{B} \in \Xi_\mu$. Then we have the following properties:

1) (Schwarz inequality)

$$|(\bar{A}\,|\,\bar{B})_\iota| \leqslant \|\bar{A}\|_\iota \|\bar{B}\|_\iota; \tag{40}$$

2) (Triangular inequality)

$$\|\bar{A} +_\iota \bar{B}\| \leqslant \|\bar{A}\|_\iota + \|\bar{B}\|_\iota; \tag{41}$$

3) (Parallelogram law)

$$\|\bar{A} +_\iota \bar{B}\|_\iota^2 + \|\bar{A} -_\iota \bar{B}\|_\iota^2 = 2\|\bar{A}\|_\iota^2 + 2\|\bar{B}\|_\iota^2. \quad (42)$$

Note that the above properties show that Ξ_μ is a normed space.

Finally, we present the generalized Pythagorean theorem.

Theorem 3 Let $\bar{A}_i \in \Xi_\mu$, $i = 1, 2, \ldots, n$ be an orthogonal set. Then,

$$\begin{aligned}&\|\bar{A}_1 +_\iota \bar{A}_2 +_\iota \ldots +_\iota \bar{A}_n\|_\iota^2 \\ &= \|\bar{A}_1\|_\iota^2 + \|\bar{A}_2\|_\iota^2 + \ldots + \|\bar{A}_n\|_\iota^2.\end{aligned} \quad (43)$$

Remark 6 All the results in this section have their corresponding results for right STP-II equivalence. Precisely speaking, we have

- Let $A, B \in \mathcal{M}_{m\times n}$. Then their right STP-II inner product is defined as

$$(A|B)_{\rm r} := (J_{t/\alpha} \otimes A \,|\, I_{t/\beta} \otimes B)_{\rm F}, \quad (44)$$

where $t = \alpha \vee \beta$.

- Let $\bar{A}_{\rm r}, \bar{B}_{\rm r} \in \Xi_\mu$. Their inner product is defined as

$$(\bar{A}_{\rm r} \,|\, \bar{B}_{\rm r})_{\rm r} := (A \,|\, B)_{\rm r}. \quad (45)$$

- The vector space $(\Xi_\mu, +_{\rm r})$ with the inner product defined by (44) is an inner product space, but not a Hilbert space.
- The norm of $\bar{A}_{\rm r} \in \Xi_\mu$ is defined as

$$\|\bar{A}_{\rm r}\|_{\rm r} := \sqrt{(\bar{A}_{\rm r} \,|\, \bar{A}_{\rm r})_{\rm r}}. \quad (46)$$

- $|(\bar{A}_{\rm r} \,|\, \bar{B}_{\rm r})_{\rm r}| \leqslant \|\bar{A}\|_{\rm r}\|\bar{B}\|_{\rm r}$; (47)
- $\|\bar{A}_{\rm r} +_{\rm r} \bar{B}_{\rm r}\| \leqslant \|\bar{A}_{\rm r}\|_{\rm r} + \|\bar{B}_{\rm r}\|_{\rm r}$; (48)
- $\|\bar{A}_{\rm r} +_{\rm r} \bar{B}_{\rm r}\|_{\rm r}^2 + \|\bar{A}_{\rm r} -_{\rm r} \bar{B}_{\rm r}\|_{\rm r}^2 = 2\|\bar{A}_{\rm r}\|_{\rm r}^2 + 2\|\bar{B}_{\rm r}\|_{\rm r}^2$. (49)

7 Metric and metric topology on Ξ_μ

Using the norm defined in previous section one sees easily that Ξ_μ is a metric space.

Theorem 4 Ξ_μ with distance

$$d_\iota(\bar{A}, \bar{B}) := \|\bar{A} -_\iota \bar{B}\|_\iota, \quad \bar{A}, \bar{B} \in \Xi_\mu \quad (50)$$

is a metric space.

Using this metric, the metric topology is obtained, which is denoted by $\mathcal{T}_{\rm d}$.

Consider

$$\mathcal{M}_\mu = \bigcup_{k=1}^{\infty} \mathcal{M}_\mu^k,$$

a natural topology can be constructed as follows:

1) each $\mathcal{M}_\mu^k$ is a clopen set;

2) on each $\mathcal{M}_\mu^k$, a natural Euclidean topology of $\mathbb{R}^{k^2\mu_y\mu_x}$ is posed.

Now we consider a natural projection $\Pr : \mathcal{M}_\mu \to \Xi_\mu$ defined by

$$\Pr(A) := \bar{A}, \quad A \in \mathcal{M}_\mu. \quad (51)$$

Similarly to STP-I equivalence case, by using projection $\Pr$, two topologies of Ξ_μ can be obtained, which are product topology $\mathcal{T}_{\rm P}$ and quotient topology $\mathcal{T}_{\rm Q}$. Again a similar argument as for STP-I equivalence, we have the following result, which is exactly the same as for STP-I equivalence case.

Theorem 5 Consider Ξ_μ. The metric topology determined by the distance d_ι is denoted by $\mathcal{T}_{\rm d}$. Then,

$$\mathcal{T}_{\rm d} \subset \mathcal{T}_{\rm Q} \subset \mathcal{T}_{\rm P}. \quad (52)$$

8 Subspaces of Ξ_μ

Consider the k-upper bounded subspace $\Xi_\mu^{[\cdot,k]} \subset \Xi_\mu$, which is defined as

$$\Xi_\mu^{[\cdot,k]} := \bigcup_{i=1}^{k} \Xi_\mu^i. \quad (53)$$

We have

Proposition 12 $\Xi_\mu^{[\cdot,k]}$ is a Hilbert space.

Proof Since $\Xi_\mu^{[\cdot,k]}$ is a finite dimensional inner space and any finite dimensional inner product space is a Hilbert space [13], the conclusion follows. □

Proposition 13 [13] Let E be an inner product space, $\{0\} \neq F \subset E$ be a Hilbert subspace.

1) For each $x \in E$ there exists a unique $y := P_{\rm F}(x) \in F$, called the projection of x on F, such that

$$\|x - y\| = \min_{z\in F} \|x - z\|. \quad (54)$$

2)

$$F^\perp := P_{\rm F}^{-1}\{0\} \quad (55)$$

is the subspace orthogonal to F.

3)

$$E = F \oplus F^{\perp}, \tag{56}$$

where $\oplus$ stands for orthogonal sum.

Using above proposition, we consider the projection: $P_{\mathrm{F}} : \Xi_\mu \to \Xi_\mu^{[\cdot,\alpha]}$. Let $\bar{A} \in \Xi_\mu^{\beta}$. Assume $\bar{X} \in \Xi_\mu^{\alpha}$, $t = \alpha \vee \beta$. Then the norm of $\bar{A} -_\iota \bar{X}$ is

$$\|\bar{A} -_\iota \bar{X}\|_\iota = \|A \otimes J_{t/\beta} - X \otimes J_{t/\alpha}\|_\iota. \tag{57}$$

Set $p = \mu_y$ $q = \mu_x$, and $k := t/\alpha$. We split A as

$$A \otimes J_{t/\beta} = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,q\alpha} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,q\alpha} \\ \vdots & \vdots & & \vdots \\ A_{p\alpha,1} & A_{p\alpha,2} & \cdots & A_{p\alpha,q\alpha} \end{bmatrix}, \tag{58}$$

where $A_{i,j} \in \mathcal{M}_{k\times k}$, $i = 1, \dots, p\alpha$; $j = 1, \dots, q\alpha$. Set

$$C := \underset{X \in \mathcal{M}_\mu^\alpha}{\operatorname{argmin}} \|A \otimes J_{t/\beta} - X \otimes J_{t/\alpha}\|_\iota. \tag{59}$$

Then, the projection $P_{\mathrm{F}} : \Xi_\mu \to \Xi_\mu^\alpha$ is defined by

$$P_{\mathrm{F}}(\bar{A}) := \bar{C}, \quad \bar{A} \in \Xi_\mu^\beta, \quad \bar{C} \in \Xi_\mu^\alpha. \tag{60}$$

It is easy to verify the following result:

Proposition 14 1) Assume $P_{\mathrm{F}}(\bar{A}) = \bar{C}$, where $A = (A_{i,j})$ is defined by (58) and $C = (c_{i,j})$ is defined by (59). Then

$$c_{i,j} = \frac{1}{k}\mathrm{tr}(A_{i,j}), \quad i = 1, \dots, p\alpha; \; j = 1, \dots, q\alpha, \tag{61}$$

where $\mathrm{tr}(A)$ is the trace of A.

2) The following orthogonality holds:

$$P_{\mathrm{F}}(\bar{A}) \perp \bar{A} - P_{\mathrm{F}}(\bar{A}). \tag{62}$$

9 Conclusions

In this paper a new matrix product, called the second STP (STP-II, or it also called MM-II product) of matrices, is proposed. To build the theory of STP-II, its properties and various geometric structures have been investigated. First, some fundamental properties are presented. Second, the STP-II caused equivalence is proposed. Based on this equivalence, the corresponding quotient space is constructed. Then the vector space structure, inner product, and the metric are all obtained. Finally, as an inner product space, some subspaces of the quotient space with orthogonal projections are considered. In own word, the quotient space has been investigated in detail.

We expect that the second STP may receive more and more applications as those of the first STP.

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