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Distributed optimal consensus of multiple double integrators under bounded velocity and acceleration

Zhirong QIU [1], Lihua XIE [1†], Yiguang HONG [2]

1.*School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798;*

2.*Key Laboratory of Systems and Control, Institute of Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

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Abstract

This paper studies a distributed optimal consensus problem for multiple double integrators under bounded velocity and acceleration. Assigned with an individual and private convex cost which is dependent on the position, each agent needs to achieve consensus at the optimum of the aggregate cost under bounded velocity and acceleration. Based on relative positions and velocities to neighbor agents, we design a distributed control law by including the integration feedback of position and velocity errors. By employing quadratic Lyapunov functions, we solve the optimal consensus problem of double-integrators when the fixed topology is strongly connected and weight-balanced. Furthermore, if an initial estimate of the optimum can be known, then control gains can be properly selected to achieve an exponentially fast convergence under bounded velocity and acceleration. The result still holds when the relative velocity is not available, and we also discuss an extension for heterogeneous Euler-Lagrange systems by inverse dynamics control. A numeric example is provided to illustrate the result.

Keywords: Distributed optimization, double integrators, bounded velocity, bounded input

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1 Introduction

For a multi-agent system, the optimal consensus problem aims to achieve the consensus among different agents, where the final consensus value is required to minimize a global cost. Usually, the global cost is given by the sum of individual costs, which are often assumed to be convex. Specifically, this problem is closely related with the distributed optimization problem arising from parallel and distributed computing, which focuses on discrete-time iteration algorithms without considering the agent dynamics. A variety of optimization algorithms have been designed in a distributed manner [1–4], and have found applications in big data [5] and distributed

†Corresponding author.
E-mail: elxie@ntu.edu.sg.

learning [6]. It is also intriguing and promising to implement these algorithms to practical systems such as robots and unmanned vehicles, in an effort to improve the performance of collective tasks.

To this aim, the corresponding continuous-time system dynamics and the constraints on the state and the input need to be considered. There have been many results considering the optimal consensus problem for continuous-time systems, with first-order integrators receiving the most attention. Based on dual decomposition, Wang and Elia proposed a saddle-point algorithm to solve an unconstrained optimization problem in [7]. The algorithm was further refined in [8] and [9] by treating the dual variable as an integral feedback to correct the consensus error, thus avoiding the communication of dual variables. The case with a common set constraint was considered in [10] by adapting the projected subgradient algorithm, and the case of different set constraints was investigated in [11]. S. Yang et al. [12] proposed a proportional-integral protocol to solve the problem. D. Mateos-Nez and J. Corts [13] dealt with coupling inequality constraints by using saddle-point dynamics with averaging of primal and dual variables. P. Yi et al. [14] investigated the case with different set constraints and coupling equality constraints for resource allocation problems. Higher-order system dynamics are also considered, e.g., the Euler-Lagrange (EL) systems [15, 16] and high-order integrators [17]. On the other hand, although the bounded control problem is very important in practice, it is not duly explored, except the recent work [18] which considered the first and second order integrators. More relevant works can be found in the multi-agent consensus literature and different approaches have been proposed, e.g., by using a low gain feedback approach [19, 20], or by tracking a virtual system with bounded control for double integrators and EL systems [21, 22], or by using a nested saturated function [23, 24].

In this paper, we study the distributed optimal consensus problem for multiple double integrators under bounded velocity and acceleration. Each agent is assigned with an individual cost, which is strongly convex and assumes Lipschitz continuous gradients. To achieve consensus at the optimum of the aggregate cost, a distributed control protocol is designed for each agent by including the following terms: a weighted sum of relative positions to its neighbors, a damping term of its velocity, a gradient descent of its individual cost, as well as an integral feedback of the relative position and the relative velocity. With the aid of quadratic Lyapunov functions, we show that the designed control law leads to an exponentially fast convergence to the global optimum under a strongly connected and weight-balanced network. If an initial estimate of the optimum is known, then the control gains can be further refined so that the optimal consensus is achieved under bounded velocity and acceleration. By using inverse dynamics control, the above result can be extended to heterogeneous EL systems with bounded velocity and control input, if the nonlinearity of each system is exactly known. Comparing with previous works, two major contributions are summarized as follows. On one hand, we not only extend the exponential convergence result in [15] for a wider class of networks which are strongly connected and weight balanced, but also achieve a similar result when the relative velocity is not available. On the other hand, an exponentially fast optimal consensus is achieved under bounded velocity and acceleration, while [18] only achieved the convergence under bounded acceleration. Although a priori knowledge of the optimum is needed to properly tune the control gains, positive definite Hessians are not required for individual costs, which is more general than the case studied in [18].

The rest of this paper is organized as follows. After briefly reviewing some preliminaries about graph theory and convex functions in Section 2, we formulate the problem under investigation in Section 3. Then we establish the optimal consensus for double-integrator multi-agent systems in Section 4.1, and extend the result to the case of EL dynamics in Section 4.2. A numeric example is provided in Section 5 to illustrate the result, and the whole work is concluded in Section 6.

Notations $\mathbb{R}$ and $\mathbb{R}_{\geqslant 0}$ respectively denote the real and nonnegative real numbers. A vector x is always viewed as a column vector, and $\mathbf{1}$ is a vector of ones with a compatible dimension. $\langle x, y\rangle$ is the standard inner product for vectors x and y, while $\|x\|$ and $\|x\|_\infty$ are respectively the 2-norm and infinity norm of x. $\mathrm{diag}\{a_1, \dots, a_n\}$ denotes a diagonal matrix with diagonal entries given by $a_1, \dots, a_n$. $\mathrm{col}\{x_1, \dots, x_n\} = [x_1', \dots, x_n']'$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively denote its smallest and largest eigenvalues. $\sum\limits_i$ denotes a summation for all possible indexes i, which similarly applies to $\max\limits_i$ and $\min\limits_i$.

2 Preliminaries

2.1 Graph theory

The communication between different agents within a multi-agent system can be modeled as a digraph $\mathcal{G}$, which consists of a node set $\mathcal{N} = \{1, \dots, N\}$ and an edge set $\mathcal{E} = \{(i, j) : i, j \in \mathcal{N}\}$ excluding self-loop (i, i). $(i, j) \in \mathcal{E}$ indicates that node j can receive information from node i, and $\mathcal{N}_i^+ = \{j : j \in \mathcal{N}, (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_i^- = \{j : j \in \mathcal{N}, (i, j) \in \mathcal{E}\}$ respectively denote the in- and out- neighbor sets of node i. $\mathcal{G}$ is said to be strongly connected if there always exists a path between two different nodes i and j, which is given as an ordered edge sequence $(i, n_1), \dots, (n_k, j)$. A non-negative matrix $A = (a_{ij}) \in \mathbb{R}_{\geqslant 0}^{N\times N}$ can be assigned as the weights on the edges, with $a_{ij} > 0$ iff $(j, i) \in \mathcal{E}$. The triplet $\{\mathcal{N}, \mathcal{E}, A\}$ completely defines a weighted digraph $\mathcal{G}$. If $A = A'$, then $\mathcal{G}$ is said to be undirected, in which case it is also said to be a connected graph if it is strongly connected. The weighted in- and out- degrees of node i are respectively given by $D_i^+ = \sum\limits_{j\in\mathcal{N}_i^+} a_{ij}$ and $D_i^- = \sum\limits_{j\in\mathcal{N}_i^-} a_{ji}$. If $D_i^+ = D_i^-$ for each i, then $\mathcal{G}$ is weight-balanced. The Laplacian matrix L can be used to examine the connectivity of $\mathcal{G}$, which is defined by $L = D_{\mathcal{G}} - A$ with $D_{\mathcal{G}}^+ = \mathrm{diag}\{D_1^+, \dots, D_N^+\}$. Note that $L\mathbf{1} = 0$, and $\mathcal{G}$ is strongly connected iff zero is a simple eigenvalue. For an undirected graph, it is connected iff the N eigenvalues of L can be rearranged as $0 = \lambda_1 < \lambda_2 \leqslant \dots \leqslant \lambda_N$. If $\mathcal{G}$ is weight-balanced, then $\mathbf{1}'L = 0$ and $\tilde{L} = \frac{1}{2}(L + L')$ is also a Laplacian matrix. Therefore, for a weight-balanced graph, it is strongly connected iff the N eigenvalues of $\tilde{L}$ can be rearranged as $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 \leqslant \dots \leqslant \tilde{\lambda}_N$.

2.2 Convex functions

A function $f : \mathbb{R}^m \to \mathbb{R}$ is convex if the inequality

$$f(ax + (1 - a)y) \leqslant af(x) + (1 - a)f(y)$$

holds for any x, y and $0 \leqslant a \leqslant 1$. If f is differentiable with gradient ∇f, then we have $f(y) - f(x) \geqslant \langle \nabla f(x), y - x\rangle$. Given a positive constant ω, f is called ω-strongly convex if

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geqslant \omega\|x - y\|^2,\ \forall x, y. \quad (1)$$

Note that (1) is equivalent to

$$f(y) \geqslant f(x) + \langle \nabla f(x), y - x\rangle + \frac{1}{2}\omega\|x - y\|^2,\ \forall x, y. \quad (1')$$

3 Problem formulation

Consider N agents with the following double-integrator dynamics:

$$\begin{cases} \dot{p}_i = v_i, \\ \dot{v}_i = u_i, \end{cases} \quad (2)$$

where $p_i, v_i, u_i \in \mathbb{R}^m$ respectively denote the position, velocity and acceleration vectors. Each agent i is assigned with an individual cost f_i, and we aim to design a distributed control law for all agents to cooperatively minimize the aggregate cost:

$$\min_{p\in\mathbb{R}^m} F(p) = \sum_i f_i(p), \quad (3)$$

or equivalently

$$\min_{p_i\in\mathbb{R}^m, i=1,\dots,N} \sum_i f_i(p_i),\ \ \text{s.t.}\ \ p_1 = \dots = p_N. \quad (3')$$

In this paper, we are concerned with those cost functions which satisfy the following assumption.

Assumption 1 For each i, f_i is differentiable and ω-strongly convex, and ∇f_i is θ-Lipschitz, namely

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leqslant \theta\|x - y\|,\ \forall x, y.$$

As a result of the strong convexity, the optimum of (3) is unique, and we denote it as p^*. Accordingly, the optimal consensus problem is equivalent to that

$$\begin{cases} \lim\limits_{t\to\infty}(p_i - p^*) = 0, \\ \lim\limits_{t\to\infty} v_i = 0,\ \ i = 1, \dots, N. \end{cases} \quad (4)$$

By the optimality condition of differential functions, we have $\sum\limits_{i=1}^{N} \nabla f_i(p^*) = 0$.

Furthermore, in practical implementation the velocity v_i and acceleration u_i are required to be bounded, respectively for safety reason and actuator limitation. Without loss of generality we consider a symmetric bound for each component as follows:

$$\|v_i\|_\infty \leqslant \bar{V},\ \ \|u_i\|_\infty \leqslant \bar{U},\ \ i = 1, \dots, N. \quad (5)$$

To design the control gains, we also need to know some a priori knowledge about the initial position and velocity, as stated in the following assumption.

Assumption 2 For each agent i, there exists a known constant R such that the optimum p^* is initially within the range of R, i.e., $\|p^* - p_i(0)\| \leqslant R$, and the initial velocity is bounded by $\|v_i(0)\|_\infty \leqslant \bar{v}_0 \leqslant \bar{V}$.

4 Main results

In this section we are to study the optimal consensus (4) of multiple double integrators under constraints (5). After establishing the exponentially fast optimal consensus result for double integrators in Section 4.1, we further discuss an extension to the case of EL dynamics by employing inverse dynamics control in Section 4.2. See the following two subsections for details.

4.1 Optimal consensus of double integrators

To solve the optimal consensus problem for agents with double-integrator dynamics (2), we design the control input u_i for each agent as follows:

$$u_i = -k_p \sum_{j\in\mathcal{N}_i^+} a_{ij}(p_i - p_j) - k_v v_i - k_{\rm g}(\mu_i + \gamma\nabla f_i(p_i)), \tag{6a}$$

$$\dot{\mu}_i = \sum_{j\in\mathcal{N}_i^+} a_{ij}[k_{\mu p}(p_i - p_j) + k_{\mu v}(v_i - v_j)],\ \mu_i(0) = 0. \tag{6b}$$

Note that the control input has a similar from as in [15], which consists of the following four terms: a weighted sum of position differences from its neighbors, a damping term of its velocity, a gradient descent of its individual cost for local minimization, as well as an integral feedback μ_i of the relative position and the relative velocity to correct the gradient differences. With different control gains for different terms, we are able to address the bounded velocity and acceleration which were not considered in [15], as well as the case when the relative velocity is not available.

Below we show that the optimal consensus problem (4) under constraint (5) can be solved with a proper selection of control gains, if the network is strongly connected and weight-balanced. Denote $\tilde{\lambda}_2$ and $\tilde{\lambda}_N$ respectively as the smallest and largest positive eigenvalues of $\tilde{L}$. To choose the control gains, let $k_v > 0$, $c_{12} \in (0, k_v)$, and $c_{23} > 0$. If the relative velocity is available ($k_{\mu v} > 0$), take k_p, $k_{\mu p}$, and $k_{\mu v}$ as

$$k_p = \beta k_{\rm g},\ \ k_{\mu p} = \frac{c_{23}}{c_3}\beta k_{\rm g},\ \ k_{\mu v} = \frac{1}{c_3}k_{\rm g}, \tag{7}$$

where $\beta > \dfrac{c_{12}}{c_{23}\tilde{\lambda}_2}$ and $c_3 > 2\dfrac{c_{23}^2 k_v}{c_{12}}$. If the relative velocity is not available ($k_{\mu v} = 0$), take $k_p, k_{\mu p}$ as

$$k_p = \beta k_{\rm g},\ \ k_{\mu p} = \frac{c_{23}}{c_3}\beta k_{\rm g}, \tag{8}$$

where $\beta > \dfrac{c_{12}}{2c_{23}\tilde{\lambda}_2}$ and $c_3 > \dfrac{c_{23}^2 k_v}{c_{12}}$. Then an exponentially fast optimal consensus can be achieved by properly choosing $k_{\rm g}$ and γ which is stated as follows.

Theorem 1 Let $\mathcal{G}$ be strongly connected and weight-balanced. Under Assumption 1 and the controller (6) with control gains given by (7) or (8), we can always find proper $k_{\rm g}$ and γ such that

1) The optimal consensus (4) is achieved at an exponential convergence rate. If the relative velocity is available, then $k_{\rm g}$ and γ need to respectively satisfy the following inequalities with $\zeta_2 = k_v - c_{12}$:

$$k_{\rm g}^2 + 2c_{23}\zeta_2 k_{\rm g} < (c_{12}c_3 - c_{23}^2 k_v)\zeta_2, \tag{9a}$$

$$k_{\rm g} \leqslant \min\Big\{\frac{c_{12}c_3}{2c_{23}} - c_{23}k_v, \frac{\tilde{\lambda}_2 c_{12}k_v(\tilde{\lambda}_2 c_{23}\beta - c_{12})}{c_{23}(\frac{c_{23}^2}{c_3}\beta + \frac{c_{12}}{2c_{23}})^2\tilde{\lambda}_N^2}\Big\}, \tag{9b}$$

$$\gamma\theta^2[c_{23}^2\zeta_2 + \frac{1}{2}k_{\rm g}(1 + \tilde{\lambda}_N)^2] < 2c_{12}\omega\zeta_2. \tag{10}$$

If the relative velocity is not available, then γ needs to satisfy (10), and $k_{\rm g}$ needs to satisfy (9a) and the following inequality

$$k_{\rm g} \leqslant \frac{\tilde{\lambda}_2 c_{12}c_3^2 k_v(2\tilde{\lambda}_2 c_{23}\beta - c_{12})}{\beta[\beta c_{23}^2\tilde{\lambda}_N^2 + 2c_3c_{12}(c_3\tilde{\lambda}_2 + c_{23}^2\tilde{\lambda}_N)]}. \tag{9b$'$}$$

2) If Assumption 2 is satisfied, then the exponentially fast optimal consensus (4) can be achieved under constraints (5). Specifically, select k_v by

$$k_v \in [\frac{\bar{\eta}}{\bar{V}}, \frac{\bar{U} - \bar{\eta}}{\bar{V}}]\ \text{ for }\ \bar{\eta} \in (0, \frac{\bar{U}}{2}], \tag{11}$$

and let $k_{\rm g}$ further satisfy that

$$k_{\rm g}\Big[(\beta\|L\| + \gamma\theta)\sqrt{\frac{c_3 - c_{23}^2}{\tilde{\Delta}}} + \sqrt{\frac{c_1\zeta_2}{\Delta}}\Big]\sqrt{2\tilde{V}_0} \leqslant \bar{\eta}, \tag{12}$$

where

$$\Delta = (c_{12}c_3 - c_{23}^2 k_v)\zeta_2 - k_{\rm g}(k_{\rm g} + 2c_{23}\zeta_2),$$
$$\tilde{\Delta} = \Delta + \tilde{\lambda}_2\beta k_{\rm g}(1 + \tilde{\lambda}_2)(c_3 - c_{23}^2),$$
$$\tilde{V}_0 = \frac{1}{2}[k_v c_{12} + c_3\gamma^2\theta^2 + \tilde{\lambda}_N\beta k_{\rm g}(1 + \tilde{\lambda}_N)$$

$$+ 2(k_g + c_{23}k_v)\gamma\theta]NR^2 + (c_{12} + c_{23}\gamma\theta)NR\bar{v}_0 + \frac{1}{2}(1 + \tilde{\lambda}_N)N\bar{v}_0^2.$$

Remark 1 Comparing with [15] which achieved an exponentially fast optimal consensus for multiple double-integrators for undirected and connected networks, we consider a wider class of networks as being strongly connected and weight-balanced, as well as the case when the relative velocity is not available. On the other hand, an exponentially fast optimal consensus is achieved under bounded velocity and acceleration, while [18] only achieved the convergence under bounded acceleration. Although a priori knowledge of the optimum is needed to properly tune the control gains, positive definite Hessians are not required for individual costs, which is more general than the case studied in [18].

Remark 2 It is readily seen that the selection of the velocity damping gain k_v is to satisfy the velocity constraint, which is only dependent on the constraints and independent from other gains. γ is the gradient gain which determines the size of individual gradient descent. Moreover, by (7) we know that c_3, c_{23} and β respectively tune the ratios between the control gains $k_{\mu v}$, $k_{\mu p}$, k_p and k_g. Specifically, the set of inequalities (9) always admits positive solution of k_g as long as $c_3 > 2\frac{c_{23}^2 k_v}{c_{12}}$, $\beta > \frac{c_{12}}{c_{23}\tilde{\lambda}_2}$, and $c_{12} \in (0, k_v)$. Once k_g has been fixed, then γ can be found by (10) for any fixed $k_g > 0$. It is also worth mentioning that although the upper bound of initial velocities $\bar{v}_0 \leqslant \bar{V}$ does not need to be known to solve k_g, a more accurate estimate of $\bar{v}_0$ enables a richer choice of k_g, as implied from (29) and the definition of $\tilde{V}_0$. For instance, if each agent starts from a static state, then $\bar{v}_0 = 0$ and $\tilde{V}_0 = \frac{1}{2}[k_v c_{12} + c_3\gamma^2\theta^2 + \tilde{\lambda}_N\beta k_g(1 + \tilde{\lambda}_N) + 2(k_g + c_{23}k_v)\gamma\theta]NR^2$. In comparison with the design process in [25] which depends on finding sufficiently small gains, the result in this paper not only provides a more flexible choice of control gains with less computation, but can also be applied to nonzero initial velocities, which is more suitable for practical implementation.

Proof of Theorem 1 The proof is divided into 3 parts as follows. In the first place we introduce some notations and rewrite the closed-loop system around the equilibrium, then we construct a quadratic Lyapunov function candidate to show the exponential convergence respectively for the cases of $k_{\mu v} > 0$ and $k_{\mu v} = 0$. Finally, we choose proper gains to satisfy the constraints (5). In the following, we let $m = 1$ since the same analysis can be applied to each component.

1) (Disagreement dynamics) Denote

$$\begin{cases} p = \mathrm{col}\{p_1, \dots, p_N\}, \ \ v = \mathrm{col}\{v_1, \dots, v_N\}, \\ \mu = \mathrm{col}\{\mu_1, \dots, \mu_N\}, \ \ u = \mathrm{col}\{u_1, \dots, u_N\}, \\ \tilde{f}(p) = \sum_i f_i(p_i), \ \ \tilde{p}^* = \mathbf{1} \otimes p^*, \\ g^* = -\nabla\tilde{f}(\tilde{p}^*) = -\mathrm{col}\{\nabla f_1(p^*), \dots, \nabla f_N(p^*)\}. \end{cases} \tag{13}$$

Furthermore, let $\rho = p - \tilde{p}^*$ and $\delta = \mu - \gamma g^*$, and combine (2) and (6) into a closed-loop system as

$$\begin{cases} \dot{\rho} = v, \\ \dot{v} = -k_p L\rho - k_v v - k_g(\delta + \gamma h(p, \tilde{p}^*)), \\ \dot{\delta} = L(k_{\mu p}\rho + k_{\mu v}v), \end{cases} \tag{14}$$

where L is the Laplacian matrix of $\mathcal{G}$, and $h(p, \tilde{p}^*) = \nabla\tilde{f}(p) - \nabla\tilde{f}(\tilde{p}^*)$. Below $h(p, \tilde{p}^*)$ will be abbreviated as h. Noticing that $\mathcal{G}$ is strongly connected and weight-balanced, we can find an orthogonal matrix $W = [w \ \tilde{W}]$ for $w = \frac{1}{\sqrt{N}}\mathbf{1}$, such that $W'\tilde{L}W = \mathrm{diag}\{0, \tilde{\Lambda}_+\}$ with $\tilde{\Lambda}_+ = \tilde{W}'\tilde{L}\tilde{W}$, and $\tilde{\lambda}_2 I \leqslant \tilde{\Lambda}_+ \leqslant \tilde{\lambda}_N I$ with $\tilde{\lambda}_2$ and $\tilde{\lambda}_N$ being respectively the smallest and largest positive eigenvalues of $\tilde{L}$. By applying the following state transformations $\hat{\rho} = \mathrm{col}\{\hat{\rho}_1, \hat{\rho}_{2:N}\} = W'\rho$, $\hat{v} = \mathrm{col}\{\hat{v}_1, \hat{v}_{2:N}\} = W'v$ and $\hat{\delta} = \mathrm{col}\{\hat{\delta}_1, \hat{\delta}_{2:N}\} = W'\delta$ to system (14) with $\hat{\rho}_1, \hat{v}_1, \hat{\delta}_1 \in \mathbb{R}^m$, we obtain that

$$\begin{cases} \dot{\hat{\rho}}_1 = \hat{v}_1, \\ \dot{\hat{v}}_1 = -k_v\hat{v}_1 - k_g\gamma w'h, \\ \dot{\hat{\delta}}_1 = 0, \end{cases} \tag{15a}$$

$$\begin{cases} \dot{\hat{\rho}}_{2:N} = \hat{v}_{2:N}, \\ \dot{\hat{v}}_{2:N} = -k_p\tilde{W}'L\tilde{W}\hat{\rho}_{2:N} - k_v\hat{v}_{2:N} - k_g\hat{\delta}_{2:N} - k_g\gamma\tilde{W}'h, \\ \dot{\hat{\delta}}_{2:N} = \tilde{W}'L\tilde{W}(k_{\mu p}\hat{\rho}_{2:N} + k_{\mu v}\hat{v}_{2:N}). \end{cases} \tag{15b}$$

Note that (15a) holds as a consequence of $w'\delta = \frac{1}{\sqrt{N}}\mathbf{1}_N'(\mu - \gamma g^*) = 0$, since we have $\mathbf{1}_N' g^* = 0$ at the equilibrium $\tilde{p}^*$, and $\mathbf{1}_N'\mu \equiv 0$ by observing that $\mathbf{1}_N'\dot{\mu} \equiv 0$ from (14) together with the initial condition $\mathbf{1}_N'\mu(0) = \sum_i \mu_i(0) = 0$.

2) (Quadratic Lyapunov function and exponential convergence) We first consider the case of $k_{\mu v} > 0$. To

prove the the convergence to the optimum, we define a quadratic Lyapunov function candidate as

$$\begin{aligned} V = &\frac{1}{2}(k_v c_{12}\|\hat{\rho}\|^2 + \|\hat{v}\|^2 + c_3\|\hat{\delta}_{2:N}\|^2) \\ &+ c_{12}\hat{\rho}'\hat{v} + c_{13}\hat{\rho}'_{2:N}\hat{\delta}_{2:N} + c_{23}\hat{v_{2:N}}'\hat{\delta}_{2:N} \\ &+ \frac{1}{2}(k_p\hat{\rho}'_{2:N}(\tilde{\Lambda}_+ + \tilde{\Lambda}_+^2)\hat{\rho}_{2:N} + \hat{v}'_{2:N}\tilde{\Lambda}_+\hat{v}_{2:N}), \end{aligned} \quad (16)$$

where $c_{13} = k_g + c_{23}k_v$. Clearly, it suffices to ensure the positive definiteness of V if $C = \begin{bmatrix} k_v c_{12} & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & c_3 \end{bmatrix}$ is positive definite, which is equivalent to that $0 < c_{12} < k_v$ and $\det(C) = c_3c_{12}(k_v - c_{12}) + 2c_{12}c_{23}c_{13} - c_{23}^2 k_v c_{12} - c_{13}^2 > 0$. Replacing $c_{13} = k_g + c_{23}k_v$ and $\zeta_2 = k_v - c_{12}$, the latter condition is exactly (9a). Noticing that $x'\tilde{W}'L\tilde{W}y = x'\tilde{\Lambda}_+y$ for any vectors x and y, $\dot{V}$ can be calculated by (15) as

$$\begin{aligned} \dot{V} = &-(c_{12}k_p - c_{13}k_{\mu p})\hat{\rho}'_{2:N}\tilde{\Lambda}_+\hat{\rho}_{2:N} \\ &- (k_v - c_{23}k_{\mu v})\hat{v}'_{2:N}\tilde{\Lambda}_+\hat{v}_{2:N} - \frac{1}{2}c_{23}k_g\|\hat{\delta}_{2:N}\|^2 \\ &+ (c_{23}k_{\mu p} + c_{13}k_{\mu v})\hat{\rho}'_{2:N}\tilde{\Lambda}_+\hat{v}_{2:N} - c_{12}k_g\hat{\rho}'_{2:N}\hat{\delta}_{2:N} \\ &- \zeta_2\|\hat{v}\|^2 - \frac{1}{2}c_{23}k_g\|\hat{\delta}_{2:N}\|^2 \\ &- k_g\gamma(\tilde{W}'h)'(c_{12}\hat{\rho} + \hat{v} + \hat{v}_{2:N} + c_{23}\hat{\delta}_{2:N}). \end{aligned} \quad (17)$$

By $k_g \leqslant \dfrac{c_{12}c_3}{2c_{23}} - c_{23}k_v$ in (9b), we get $c_{23}k_g \leqslant \dfrac{1}{2}c_3c_{12} - c_{23}^2k_v$; together with $c_3 > 2\dfrac{c_{23}^2k_v}{c_{12}}$, we get $c_{13} \leqslant \dfrac{c_{12}c_3}{2c_{23}}$, and hence

$$\begin{cases} c_{13}k_{\mu p} = c_{13}\dfrac{c_{23}}{c_3}k_p \leqslant \dfrac{1}{2}c_{12}k_p, \\ c_{23}k_{\mu v} = c_{23}\dfrac{1}{c_3}k_g < c_{23}\dfrac{c_{12}}{c_3}k_g < \dfrac{1}{2}k_v. \end{cases} \quad (18)$$

By combining (18) and Assumption 1 into (17), we can estimate the upper bound of $\dot{V}$ as

$$\dot{V} \leqslant -\frac{1}{2}\xi_1'\tilde{S}_1\xi_1 - \frac{1}{2}\xi_2'\tilde{S}_2\xi_2, \quad (19)$$

where

$$\xi_1 = [\|\hat{\rho}_{2:N}\|, \|\hat{v}_{2:N}\|, \|\hat{\delta}_{2:N}\|]', \quad \xi_2 = [\|\hat{\rho}\|, \|\hat{v}\|, \|\hat{\delta}_{2:N}\|]',$$

$$\tilde{S}_1 = \begin{bmatrix} c_{12}k_p\tilde{\lambda}_2 & -(c_{23}k_{\mu p}+c_{13}k_{\mu v})\tilde{\lambda}_N & -c_{12}k_g \\ -(c_{23}k_{\mu p}+c_{13}k_{\mu v})\tilde{\lambda}_N & c_{12}k_p\tilde{\lambda}_2 & 0 \\ -c_{12}k_g & 0 & c_{23}k_g \end{bmatrix},$$

$$\tilde{S}_2 = \begin{bmatrix} z2c_{12}k_g\gamma\omega & -k_g\gamma\theta(1+\tilde{\lambda}_N) & -k_g c_{23} \\ -k_g\gamma\theta(1+\tilde{\lambda}_N) & 2\zeta_2 & 0 \\ -k_g c_{23} & 0 & c_{23}k_g \end{bmatrix}.$$

To show that $\dot{V}$ is negative definite, it is enough to show that $\tilde{S}_1$ is positive semi-definite and $\tilde{S}_2$ is positive definite, which is equivalent to that $\det\tilde{S}_1 \geqslant 0$ and $\det\tilde{S}_2 > 0$ by direct computation. To be detailed,

$$\begin{aligned} &\det(\tilde{S}_1) \\ &= k_g[c_{12}k_v\tilde{\lambda}_2(c_{23}k_p\tilde{\lambda}_2 - c_{12}k_g) - c_{23}(c_{23}k_{\mu p} + c_{13}k_{\mu v})^2\tilde{\lambda}_N^2] \\ &= k_g^2[c_{12}k_v\tilde{\lambda}_2(c_{23}\beta\tilde{\lambda}_2 - c_{12}) - c_{23}(\frac{c_{23}^2}{c_3}\beta + \frac{c_{13}}{c_3})^2\tilde{\lambda}_N^2k_g] \\ &\geqslant k_g^2[c_{12}k_v\tilde{\lambda}_2(c_{23}\beta\tilde{\lambda}_2 - c_{12}) - c_{23}(\frac{c_{23}^2}{c_3}\beta + \frac{c_{12}}{2c_{23}})^2\tilde{\lambda}_N^2k_g] \\ &\geqslant 0, \end{aligned} \quad (20)$$

where the two inequalities are respectively due to $c_{13} \leqslant \dfrac{c_{12}c_3}{2c_{23}}$ and (9b). On the other hand, it can also be checked that $\det\tilde{S}_2 > 0$ when condition (10) is satisfied, and we conclude the exponential convergence.

When $k_{\mu v} = 0$, we can proceed along the same line as in the above to obtain a similar estimate to (19), where $\tilde{S}_1$ is now given by $\tilde{S}_1 = \begin{bmatrix} 2c_{12}k_p\tilde{\lambda}_2 & -c_{23}k_{\mu p}\tilde{\lambda}_N & -c_{12}k_g \\ -c_{23}k_{\mu p}\tilde{\lambda}_N & 2c_{12}k_p\tilde{\lambda}_2 & -k_g \\ -c_{12}k_g & -k_g & c_{23}k_g \end{bmatrix}$. It can be easily checked that the positive semi-definiteness of $\tilde{S}_1$ is guaranteed by (9b′) with $\beta > \dfrac{c_{12}}{2c_{23}\tilde{\lambda}_2}$. Moreover, since $k_{\mu v} = 0$, (18) holds naturally without the condition $k_g \leqslant \dfrac{c_{12}c_3}{2c_{23}} - c_{23}k_v$, and it suffices to require that $c_3 > \dfrac{c_{23}^2k_v}{c_{12}}$ so that (9a) admits positive solutions of k_g. Therefore, the exponential convergence can be preserved if the other conditions remain the same.

3) (Optimal consensus under constraint (5)) We first discuss the bounded velocity requirement. Recall that we have assumed that v_i is a scalar, whose dynamics is given by (14) as $\dot{v}_i = -k_v v_i + \eta_i$, with η_i being the ith entry of $\eta = -k_pL\rho - k_g(\delta + \gamma h)$. Assume that $|\eta_i| \leqslant \bar{\eta}$, then it is easy to show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_i|^2 = v_i(-k_v v_i + \eta_i \leqslant |v_i|(-k_v|v_i| + \bar{\eta}) \leqslant 0,$$

when $|v_i| \geqslant \dfrac{\bar{\eta}}{k_v}$. As a result, if $|v_i(0)| \leqslant \bar{V}$ with $\dfrac{\bar{\eta}}{k_v} \leqslant \bar{V}$, then it follows that $|v_i(t)| \leqslant \bar{V}$ for $t > 0$. On the other hand, to satisfy the bound of control input, it suffices to require that $k_v\bar{V} + \bar{\eta} \leqslant \bar{U}$. In summary, we need the following condition to hold:

$$\begin{cases} k_v\bar{V} + \bar{\eta} \leqslant \bar{U}, \\ 0 < \bar{\eta} \leqslant k_v\bar{V}, \end{cases} \tag{21}$$

which is equivalent to (11). Based on the above discussion, next we shall estimate $\|\eta\|_\infty$.

To be detailed, note that the upper bound of η is dependent on that of ρ and δ, which can be estimated by employing the quadratic Lyapunov function in step 2). Actually, $\dot{V}(t) \leqslant 0$ implies that

$$\begin{aligned} V(t) &\leqslant V(0) \\ &\leqslant \frac{k_vc_{12}}{2}\|\rho_0\|^2 + c_{12}\|\rho_0\|\,\|v_0\| + \frac{1}{2}\|v_0\|^2 \\ &\quad + \frac{c_3}{2}\|\delta_0\|^2 + (c_{13}\|\rho_0\| + c_{23}\|v_0\|)\|\delta_0\| \\ &\quad + \frac{\tilde{\lambda}_N}{2}[k_p(1+\tilde{\lambda}_N)\|\rho_0\|^2 + \|v_0\|^2] \leqslant \tilde{V}_0, \end{aligned} \tag{22}$$

where $\tilde{V}_0$ is defined in (12), and the subscript 0 denotes the corresponding initial values of ρ, v, and δ, and we used the facts that $\|\delta_0\| = \|\mu_0 - \gamma g^*\| = \|\gamma g^*\| \leqslant \gamma\theta\|\rho_0\|$, $\|v_0\|^2 \leqslant N\bar{v}_0^2$ and $\|\rho_0\|^2 \leqslant NR^2$. To find an upper bound for ρ and δ, note that

$$\begin{cases} \frac{1}{2}[c_{12}\zeta_2 - \frac{(c_{13}-c_{12}c_{23})^2}{c_3 - c_{23}^2} + \tilde{\lambda}_Nk_p(1+\tilde{\lambda}_N)]\|\rho\|^2 \leqslant V(t), \\ \frac{1}{2}[c_3 - \frac{c_{13}^2}{k_vc_{12}} - \frac{(k_vc_{23}-c_{13})^2}{k_v\zeta_2}]\|\delta\|^2 \leqslant V(t). \end{cases} \tag{23}$$

Combining (22) and (23), we obtain that $\|\rho\| \leqslant \sqrt{\frac{2(c_3 - c_{23}^2)\tilde{V}_0}{\tilde{\varDelta}}}$ and $\|\delta\| \leqslant \sqrt{\frac{2c_1\zeta_2\tilde{V}_0}{\varDelta}}$ after plugging $c_{13} = k_g + c_{23}k_v$ and $k_p = \beta k_g$, where $\tilde{\varDelta}$ and $\varDelta$ are defined in (12). Now we can find an upper bound of $\eta = -k_pL\rho - k_g(\delta + \gamma h)$ as

$$\begin{aligned} \|\eta\|_\infty &\leqslant k_g[\beta\|L\|\,\|\rho\| + \|\delta\| + \gamma\theta\|\rho\|] \\ &\leqslant k_g[(\beta\|L\| + \gamma\theta)\sqrt{\frac{c_3 - c_{23}^2}{\tilde{\varDelta}}} + \sqrt{\frac{c_1\zeta_2}{\varDelta}}]\sqrt{2\tilde{V}_0}. \end{aligned} \tag{24}$$

Therefore, to guarantee an exponentially fast consensus (4) under the constraint (5), it suffices to further require k_v and k_g respectively satisfy (11) and (12). □

If the network is undirected and connected, then we can find simpler conditions to select the control gains by employing a simpler Lyapunov function. In this case, given $k_v, \gamma > 0$, $c_{12} \in (0, k_v)$, and $b_{23}, c_3 > 0$, select k_p, $k_{\mu p}$, and $k_{\mu v}$ as

$$\begin{cases} k_p = \frac{c_{12}}{c_3}b_{23}k_g, \quad k_{\mu p} = \frac{c_{12}}{c_3}k_g, \\ k_{\mu v} = \begin{cases} \frac{1}{c_3}k_g, & \text{if the relative velocity is available}, \\ 0, & \text{otherwise}. \end{cases} \end{cases} \tag{25}$$

Furthermore, define

$$\begin{cases} \zeta_2 = k_v - c_{12}, \quad \zeta_{12} = \frac{c_{12}}{c_3}b_{23}\lambda_N + (b+1)\gamma\theta, \\ \zeta_{23} = (\frac{b_{23}}{c_3}\lambda_N + \frac{1}{b_{23}})k_g + k_v. \end{cases} \tag{26}$$

Then an exponentially fast optimal consensus can be achieved by properly choosing k_g as follows.

Theorem 2 Let $\mathcal{G}$ be undirected and connected, and denote λ_2 and λ_N respectively as the smallest and largest positive eigenvalues of L. Under Assumption 1 and the controller (6) with control gains given by (25), we can always find proper γ such that

1) The optimal consensus (4) is achieved at an exponential convergence rate.

If the relative velocity is available, then k_g needs to satisfy the following inequalities for a proper selected $b > 0$:

$$k_g \leqslant \frac{c_3}{b_{23}\lambda_N}k_v, \tag{27a}$$

$$\begin{aligned} &k_g\gamma(4c_{12}\omega - bb_{23}\gamma\theta^2)\zeta_2 \\ &> \zeta_{12}^2k_g^2 + bb_{23}\gamma\zeta_{23}(c_{12}\omega\zeta_{23} + \zeta_{12}k_g\theta). \end{aligned} \tag{27b}$$

If the relative velocity is not available, then k_g needs to satisfy the following inequalities:

$$k_g \leqslant \min\{b_{23}, \sqrt{\frac{c_{12}c_3}{\lambda_N}\zeta_2}\}, \tag{28a}$$

$$\begin{aligned} &k_g\gamma(4c_{12}\omega - bb_{23}\gamma\theta^2)\zeta_2 \\ &> \zeta_{12}^2k_g^2 + bb_{23}\gamma c_{12}\omega(\frac{k_g}{b_{23}} + k_v)^2 + k_g\zeta_{12}bb_{23}\gamma\theta(\frac{k_g}{b_{23}} + k_v). \end{aligned} \tag{28b}$$

2) If Assumption 2 is satisfied, then the exponentially fast optimal consensus (4) can be achieved under constraints (5). Specifically, select k_v by (11), and let k_g

further satisfy that

$$k_g[(\frac{c_{12}b_{23}}{c_3}\lambda_N + \gamma\theta)\sqrt{\frac{1}{c_{12}\zeta_2}} + \sqrt{\frac{\lambda_N}{c_3}}]\sqrt{2\bar{V}_0} \leqslant \bar{\eta} \quad (29)$$

for $k_{\mu v} > 0$, and

$$k_g[(\frac{c_{12}b_{23}}{c_3}\lambda_N+\gamma\theta)\sqrt{c_3}+\sqrt{c_{12}\zeta_2\lambda_N}]\sqrt{\frac{2(\bar{V}_0+k_g\gamma\theta NR^2)}{c_{12}\zeta_2 c_3 - k_g^2\lambda_N}} \leqslant \bar{\eta} \quad (29')$$

for $k_{\mu v}=0$, where $\bar{V}_0=\frac{1}{2}(k_v c_{12}+\frac{c_3\gamma^2\theta^2}{\lambda_2})NR^2+c_{12}NR\bar{v}_0+\frac{1}{2}N\bar{v}_0^2$.

Remark 3 In comparison with Theorem 1, it can be seen that there is less restriction in choosing the control gains. Actually, the ratios between the control gains $k_{\mu v}$, k_p and k_g which are determined respectively by c_3 and b_{23}, can now be chosen arbitrarily. This is also the case for the gradient gain γ, and we only need to solve the inequalities with respect to k_g. With a properly selected b, positive k_g can always be found to satisfy (27b) or (28b), since either one can be approximated by the following inequality with sufficiently small $b > 0$:

$$4\gamma c_{12}\omega(k_v - c_{12}) > k_g(\frac{c_{12}}{c_3}b_{23}\lambda_N + \gamma\theta)^2. \quad (30)$$

Furthermore, it is worthwhile mentioning that by using a Lyapunov function with negative semi-definite derivative as shown in the subsequent proof, the required condition (29) to satisfy the bounded acceleration for $k_{\mu v} > 0$ is much simpler than the corresponding condition (12) in Theorem 1.

Remark 4 When the relative velocity is available, i.e., $k_{\mu v} > 0$, we may select k_g to accelerate the convergence when the other parameters haven been decided. Actually, in the proof we employ a quadratic Lyapunov function V to show the exponential convergence $\dot{V} \leqslant -\frac{\lambda_{\min}(S)}{\lambda_{\max}(\Gamma)}V$, where S is defined in (34) and $\Gamma = \frac{1}{2}\begin{bmatrix} k_v c_{12} & c_{12} & 0 \\ c_{12} & 1+b & bb_{23} \\ 0 & bb_{23} & bb_{23}^2 + \frac{c_3}{\lambda_2}\end{bmatrix}$. In this case, given an error bound ϵ, an upper bound of the time T for all agents reaching the ϵ-neighborhood of the optimum is found as $T = \frac{C}{\alpha}\ln\frac{1}{\epsilon}$ with $C > 0$ and $\alpha = \frac{\lambda_{\min}(S)}{\lambda_{\max}(\Gamma)}$. Hence, a faster convergence can be expected with a larger α. Since k_g only appears in S which is linearly dependent on k_g, we can find the optimal k_g by solving the following semidefinite programing problem:

$$\begin{aligned} &\max\ \lambda, \\ &\text{s.t. } S \geqslant \lambda I, \\ &\qquad k_g \in \Omega \triangleq \{k_g : k_g \text{ satisfies (27) and (29)}\}. \end{aligned} \quad (31)$$

To obtain a larger α, the other parameters can be determined by a grid search [26]. Still, it should be pointed out that the design of control gains depends on the specific choice of V, and the corresponding α only provides a lower bound of the convergence rate, which may be conservative.

Proof of Theorem 2 The proof follows a similar line to the proof of Theorem 1, by using different Lyapunov functions for different cases. Noticing that $\mathcal{G}$ is undireted and connected, we can always find an orthogonal matrix $W = [w\ \tilde{W}]$ for $w = \frac{1}{\sqrt{N}}\mathbf{1}$, such that $W'LW = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$ with $0 < \lambda_2 \leqslant \ldots \leqslant \lambda_N$. By multiplying W' to ρ, v and δ respectively, we can similarly obtain the transformed disagreement dynamics as (15), with $\tilde{W}'L\tilde{W}$ in (15b) replaced by $\Lambda_+ = \mathrm{diag}\{\lambda_2, \ldots, \lambda_N\}$. Below we discuss the cases of $k_{\mu v} > 0$ and $k_{\mu v} = 0$ respectively.

1) $k_{\mu v} > 0$. To prove the convergence to the optimum, we define a quadratic Lyapunov function candidate as

$$\begin{aligned} V = &\frac{1}{2}c_1\|\hat{\rho}\|^2 + c_{12}\hat{\rho}'\hat{v} + \frac{1}{2}\|\hat{v}\|^2 + \frac{1}{2}b\|\hat{v}_{2:N} + b_{23}\hat{\delta}_{2:N}\|^2 \\ &+ \frac{1}{2}c_3\hat{\delta}_{2:N}'(\Lambda_+)^{-1}\hat{\delta}_{2:N}, \end{aligned} \quad (32)$$

where we let $c_1 = c_{12}k_v$. Similar to (19), we can find an upper bound of $\dot{V}$ as

$$\begin{aligned} \dot{V} \leqslant &-c_{12}k_g\gamma\omega\|\hat{\rho}\|^2 - \zeta_2\|\hat{v}\|^2 - bb_{23}k_g\|\hat{\delta}_{2:N}\|^2 \\ &+ \zeta_{12}k_g\|\hat{\rho}\|\,\|\hat{v}\| + bb_{23}k_g\gamma\theta\|\hat{\rho}\|\,\|\hat{\delta}_{2:N}\| \\ &+ bb_{23}\zeta_{23}\|\hat{v}\|\,\|\hat{\delta}_{2:N}\| \\ \triangleq &-\xi'S\xi, \end{aligned} \quad (33)$$

where ζ_2, ζ_{12}, ζ_{23} have been defined in (26), $\xi = [\|\hat{\rho}\|, \|\hat{v}\|, \|\hat{\delta}_{2:N}\|]'$, and $S = [s_{ij}] \in \mathbb{R}^{3\times 3}$ is a symmetric matrix whose entries are given by

$$\begin{cases} s_{11} = c_{12}k_g\gamma\omega,\ \ s_{22} = \zeta_2,\ \ s_{33} = bb_{23}k_g, \\ s_{12} = -\frac{1}{2}\zeta_{12}k_g,\ \ s_{13} = -\frac{1}{2}bb_{23}k_g\gamma\theta,\ \ s_{23} = -\frac{1}{2}bb_{23}\zeta_{23}. \end{cases} \quad (34)$$

Obviously, to ensure the positive definiteness of V, it suffices to let $c_1 = c_{12}k_v > c_{12}^2$, which is equivalent to

$$k_v > c_{12}. \tag{35}$$

Furthermore, to ensure the negative definiteness of $\dot V$, it suffices to require that S be positive definite, which is equivalent to $S_k > 0$ with S_k being the kth leading principal minor of S for $k = 1,2,3$. Direct computation shows that $S_1 = s_{11}$, $S_2 = s_{11}s_{22} - s_{12}^2$, and

$$S_3 = s_{33}[S_2 - \frac{1}{4}bb_{23}(c_{12}\gamma\omega\zeta_{23}^2 + \zeta_{12}k_{\rm g}\zeta_{23}\gamma\theta + k_{\rm g}\zeta_2(\gamma\theta)^2)].$$

Therefore, with $\zeta_2 > 0$, the negative definiteness of $\dot V$ is equivalent to $S_3 > 0$, which is ensured by (27b).

To satisfy the velocity constraint, the same argument in the proof of Theorem 1 can be applied. On the other hand, to meet the acceleration bound, we need to estimate the upper bound of $\eta = -k_pL\rho - k_{\rm g}(\delta + \gamma h)$. In this case, we will use a different Lyapunov function by letting $V = V_1$ with $b = 0$. It can be checked that V_1 is positive defintie with the condition (35), and it can be obtained from (33) that $\dot V_1 \leqslant -c_{12}k_{\rm g}\gamma\omega\|\hat\rho\|^2 - \zeta_2\|\hat v\|^2 + (k_p\lambda_N + k_{\rm g}\gamma\theta)\zeta_{12}\|\hat\rho\|\,\|\hat v\|$. Therefore, $\dot V_1 \leqslant 0$ if the condition (27b) holds, which follows that

$$\begin{aligned} V_1(t) &\leqslant V_1(0)\\ &\leqslant \frac{k_vc_{12}}{2}\|\rho_0\|^2 + c_{12}\|\rho_0\|\,\|v_0\| + \frac{1}{2}\|v_0\|^2 + \frac{c_3}{2\lambda_2}\|\delta_0\|^2\\ &\leqslant \bar V_0, \end{aligned} \tag{36}$$

where $\bar V_0$ is defined in (29), and the subscript 0 denotes the corresponding initial values of ρ, v, and δ, and we used the facts that $\|\delta_0\| = \|\mu_0 - \gamma g^*\| = \|\gamma g^*\| \leqslant \gamma\theta\|\rho_0\|$, $\|v_0\|^2 \leqslant N\bar v_0^2$ and $\|\rho_0\|^2 \leqslant NR^2$. In addition, it can be seen that

$$\begin{aligned} V_1(t) &= \frac{1}{2}[\|\hat v + c_{12}\hat\rho\|^2 + c_{12}\zeta_2\|\rho\|^2 + c_3\hat\delta_{2:N}'(\Lambda_+)^{-1}\hat\delta_{2:N}]\\ &\geqslant \frac{1}{2}\|\hat v\|^2 + \frac{c_3}{2\lambda_N}\|\delta\|^2. \end{aligned} \tag{37}$$

Combining (36) and (37), we obtain that $\|\rho\| \leqslant \sqrt{\dfrac{2\bar V_0}{c_{12}\zeta_2}}$ and $\|\delta\| \leqslant \sqrt{\dfrac{2\lambda_N\bar V_0}{c_3}}$. By substituting $k_p = \dfrac{c_{12}}{c_3}b_{23}k_{\rm g}$, we can find an upper bound of $\eta = -k_pL\rho - k_{\rm g}(\delta - \gamma h)$ as

$$\begin{aligned} \|\eta\|_\infty &\leqslant k_{\rm g}[\frac{c_{12}b_{23}}{c_3}\lambda_N\|\rho\| + \|\delta\| + \gamma\theta\|\rho\|]\\ &\leqslant k_{\rm g}\Big[(\frac{c_{12}b_{23}}{c_3}\lambda_N + \gamma\theta)\sqrt{\frac{1}{c_{12}\zeta_2}} + \sqrt{\frac{\lambda_N}{c_3}}\Big]\sqrt{2\bar V_0}. \end{aligned} \tag{38}$$

Therefore, to guarantee an exponentially fast consensus (4) under the constraint (5), we can first select $\bar\eta \in (0, \bar U/2]$ and $k_v \in [\bar\eta/\bar V, (\bar U - \bar\eta)/\bar V]$, then for fixed $\gamma, b_{23}, c_{12} \in (0, k_v), c_3$, obtain $k_{\rm g}$ by solving the inequalities (27) and (29) with a properly selected b. The other control gains now can be found by (25).

2) $k_{\mu v} = 0$. In this case, we revise the Lyapunov candidate (32) by adding an extra term $k_{\rm g}\hat\rho_{2:N}'\hat\delta_{2:N}$ to V, namely $\tilde V = V + k_{\rm g}\hat\rho_{2:N}'\hat\delta_{2:N}$. It is easy to verify that the positive definiteness of $\tilde V$ is satisfied with

$$c_{12}\zeta_2\frac{c_3}{\lambda_N} > k_{\rm g}^2, \tag{39}$$

which follows as a consequence of (28a). Moreover, if $k_{\rm g} \leqslant b_{23}$, then by proceeding as (33) we get $\dot{\tilde V} \leqslant -\xi'\tilde S\xi$, where $\tilde S = [\tilde s_{ij}]$ is a symmetric matrix with $\tilde s_{ij} = s_{ij}$, except that $\tilde s_{23} = -\dfrac{1}{2}bb_{23}\tilde\zeta_{23}$ with $\tilde\zeta_{23} = \dfrac{k_{\rm g}}{b_{23}} + k_v$. Accordingly, the condition (27b) to guarantee the exponentially fast optimal consensus is revised as (28b). Now we have $\tilde V(t) \leqslant \bar V_0 + k_g\gamma\theta NR^2$ and that

$$\begin{cases} \|\rho\| \leqslant \sqrt{\dfrac{2(\bar V_0 + k_g\gamma\theta NR^2)}{c_{12}\zeta_2 - k_g^2\lambda_N/c_3}},\\ \|\delta\| \leqslant \sqrt{\dfrac{2\lambda_N(\bar V_0 + k_g\gamma\theta NR^2)}{c_3/\lambda_N - k_g^2/(c_{12}\zeta_2)}}, \end{cases} \tag{40}$$

as a result of

$$\begin{cases} (\dfrac{c_3}{\lambda_N} - \dfrac{k_g^2}{c_{12}\zeta_2})\|\hat\delta\|^2 \leqslant 2\tilde V(t),\\ (c_{12}\zeta_2 - \dfrac{k_g^2}{c_3/\lambda_N})\|\hat\rho\|^2 \leqslant 2\tilde V(t). \end{cases} \tag{41}$$

Therefore, in order to achieve the optimal consensus under bounded input, it suffices to revise the corresponding condition as (29′). □

4.2 Optimal consensus with EL dynamics

In this section, we shall study the optimal consensus problem for heterogeneous EL systems under bounded velocity and control input. Specifically, consider N agents with the following EL dynamics:

$$M_i(q_i)\ddot q_i + C_i(q_i, \dot q_i)\dot q_i = \tau_i, \quad i = 1, \dots, N, \tag{42}$$

where $q_i, \dot{q}_i, \ddot{q}_i \in \mathbb{R}^m$ respectively denote the generalized position, velocity and acceleration vectors; $M_i \in \mathbb{R}^{m\times m}$ is the positive definite inertia matrix dependent on the position; $C_i(q_i, \dot{q}_i)\dot{q}_i$ is the Coriolis and centripetal forces; τ_i is the control force. It is common to adopt the following asssumption for M_i and C_i [27].

Assumption 3 There exist positive constants M and C such that $\|M_i(q_i)\| \leqslant M$ and $\|C_i(q_i, \dot{q}_i)\| \leqslant C\|\dot{q}_i\|$.

As the case of double integrators, we assign each agent i with a cost function f_i which is dependent on the generalized position q_i, and aim to achieve the consensus at the minimum of the aggregate cost. Under Assumption 1 we denote the unique optimum $q^* = \arg\min_{q\in\mathbb{R}^m} \sum_i f_i(q)$, and the corresponding optimal consensus problem is given by

$$\begin{cases} \lim\limits_{t\to\infty}(q_i - q^*) = 0, \\ \lim\limits_{t\to\infty} \dot{q}_i = 0, \quad i = 1, \dots, N. \end{cases} \tag{4'}$$

Moreover, we are concerned with the following constraints on the generalized velocity and control input:

$$\|\dot{q}_i\|_\infty \leqslant \bar{V}, \quad \|\tau_i\|_\infty \leqslant \bar{U}, \quad i = 1, \dots, N. \tag{5'}$$

A similar assumption about the initial state is made below.

Assumption 4 For each agent i, there exists a known constant R such that the optimum q^* is initially within the range of R, i.e., $\|q^* - q_i(0)\| \leqslant R$, and the initial velocity is bounded by $\|\dot{q}_i(0)\|_\infty \leqslant \bar{v}_0 \leqslant \bar{V}$.

Assuming that M_i and C_i can be accurately obtained, we are ready to solve the optimal consensus problem (4′) for EL dynamical systems under constraints (5′) by adopting inverse dynamics control. Let τ_i be given by

$$\tau_i = M_i(q_i)u_i + C_i(q_i, \dot{q}_i)\dot{q}_i, \tag{43}$$

then the original EL system (42) is reduced to the double-integrator system (2). Clearly, if we design u_i by

$$u_i = -k_p \sum_{j\in\mathcal{N}_i^+} a_{ij}(q_i - q_j) - k_v\dot{q}_i - k_g(\mu_i + \gamma\nabla f_i(q_i)) \tag{44}$$

with $\dot{\mu}_i = \sum_{j\in\mathcal{N}_i^+} a_{ij}[k_{\mu p}(q_i - q_j) + k_{\mu v}(\dot{q}_i - \dot{q}_j)]$ and $\dot{\mu}_i(0) = 0$, then we have $q_i \equiv p_i$ and $\dot{q}_i \equiv v_i$ for identical initial values. Therefore, the exponentially fast optimal consensus can be obtained under the exactly same conditions in Theorem 1 or 2. To further satisfy the constraint (5′), we let $\|\dot{q}_i\|_\infty \leqslant \hat{V} \leqslant \bar{V}$ and $\|\eta_i\|_\infty \leqslant \bar{\eta}$ for $\eta_i = -k_p \sum_{j\in\mathcal{N}_i^+} a_{ij}(q_i - q_j) - k_g(\mu_i + \gamma\nabla f_i(q_i))$. Proceeding along a similar line as in step 3) of the proof of Theorem 1 and invoking Assumption 3, we know that the constraints on $\dot{q}_i$ and τ_i are met with $0 < \bar{\eta} \leqslant k_v\hat{V}$ and $M(k_v\hat{V} + \bar{\eta}) + C\hat{V}^2 \leqslant \bar{U}$, which is equivalent to that

$$\begin{cases} 0 < \bar{\eta} \leqslant \dfrac{\bar{U} - C\hat{V}^2}{2M}, \\ \dfrac{\bar{\eta}}{\hat{V}} \leqslant k_v \leqslant \dfrac{\bar{U} - C\hat{V}^2 - M\bar{\eta}}{M\hat{V}}. \end{cases} \tag{45}$$

Moreover, η_i can be upper bounded similarly as in Theorem 1 or 2, if the initial range of optimum is known, and the initial velocity is bounded by $\|\dot{q}_i(0)\|_\infty \leqslant \hat{V}$. Consequently, the optimal consensus (4′) can be achieved under the constraint (5′). We summarize the above discussion in the following theorem.

Theorem 3 Let Assumptions 1 and 3 hold, and the controller be given by (43) and (44). Furthermore, let Assumption 4 hold with $\bar{v}_0 \leqslant \hat{V} < \max\{\sqrt{\bar{U}/C}, \bar{V}\}$. Choose k_v by (45), then for both cases of $k_{\mu v} > 0$ and $k_{\mu v} = 0$, the exponentially fast optimal consensus (4′) can be achieved under constraints (5′) by properly selecting the other control gains as follows.

1) If $\mathcal{G}$ is strongly connected and weight-balanced, then select the control gains as in Theorem 1.

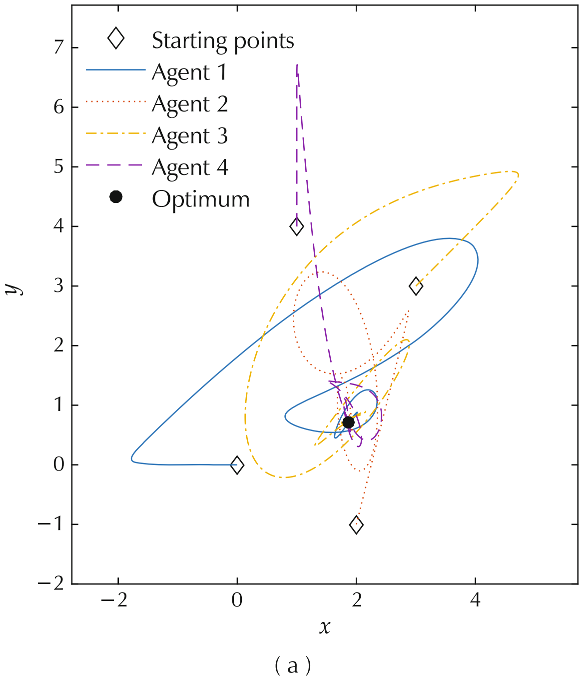
2) If $\mathcal{G}$ is undirected and connected, then select the control gains as in Theorem 2.

Remark 5 Comparing Theorem 3 with Theorem 1 or 2, we find that the only difference is the inequality (45) about k_v and $\bar{\eta}$ due to the different control τ_i, which requires a possibly lower bound of velocity $\hat{V} \leqslant \bar{V}$ due to the additional term $C_i\dot{q}_i$. However, once k_v and $\bar{\eta}$ have been determined, the other control gains can be found similarly as in Theorem 1 or 2.

5 Numerical example

We consider a 4-agent system with double-integrator dynamics (2) under a connected network. The constraints in (5) are given by $\bar{V} = \bar{U} = 5$, respectively corresponding to the maximum velocity and acceleration of 5 m/s and 5 m/s^2. Assume that the 4 agents are connected as a circle with weight 0.1 on each edge, which follows that $\lambda_2 = 0.2$ and $\lambda_N = 0.4$. Furthermore, denote the position of the ith agent as $p_i = [x_i, y_i]'$, with the corresponding individual cost $f_i = a_i(x_i - x_i(0))^2 + b_i(y_i - y_i(0))^2$, where the coefficients are respectively given by $a_i = 0.3, 0.6, 0.9, 0.6$, $b_i = 0.6, 0.6, 0.3, 0.3$, and $[x_i(0), y_i(0)]'$ is the initial position. Let the agents respectively start from $[0,0]'$, $[2,-1]'$, $[3,3]'$ and $[1,4]'$ with the initial velocity given by $[-1,0]'$, $[0.5,2]'$, $[1,1]'$ and $[0,1.5]'$, then it can be inferred that $R = 5.39$ and $\bar{v}_0 = 2$. In light of (1′) we can also get $\omega = \min_i\{a_i, b_i\} = 0.3$, and $\theta = 2\max_i\{a_i, b_i\} = 1.8$. To show the efficacy of the control protocol (6), in the following we shall select the control gains as Remark 4, and discuss the influence of the gradient gain γ on the convergence rate and trajectory.

To be detailed, we fix $k_v = 0.5$ for $\bar{\eta} = 2.5$, $c_3 = c_{12} = \frac{k_v}{2}$ and $b_{23} = 0.1$, and respectively select $\gamma = 0.1, 0.4$ and 0.8. A grid search of b is conducted in $[0.01, 1]$ with an increment of 0.01 in each loop, and k_g is found by solving the problem (31) in Remark 4. As demonstrated in Figs. 1–3, in all cases the consensus can be achieved exponentially fast at the optimum $[1.88, 0.71]'$ under bounded velocity and acceleration. Furthermore, it also shows a trade-off between convergence rate and trajectory oscillation: a large γ leads to a smooth trajectory with slow convergence, while a small γ leads to fast convergence with an oscillating trajectory.



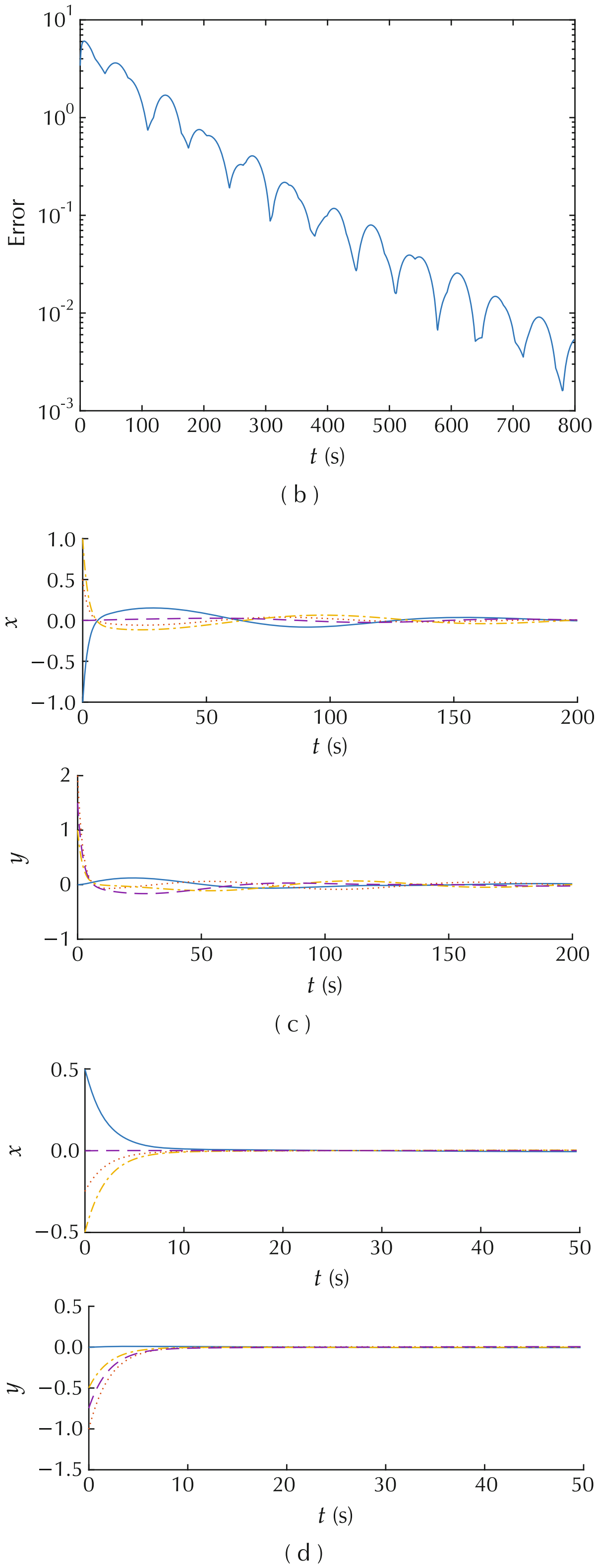
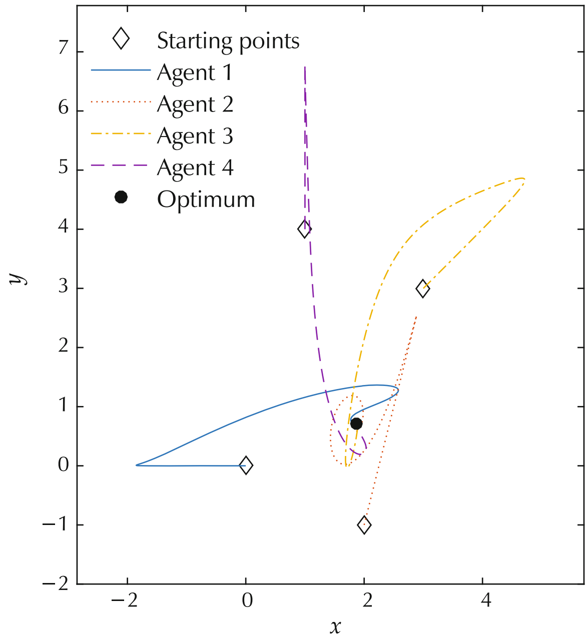
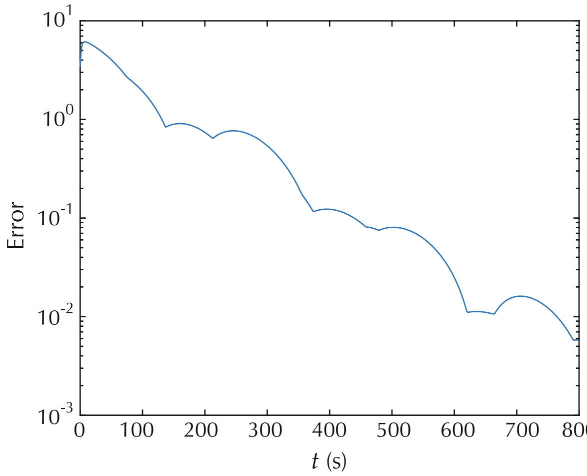


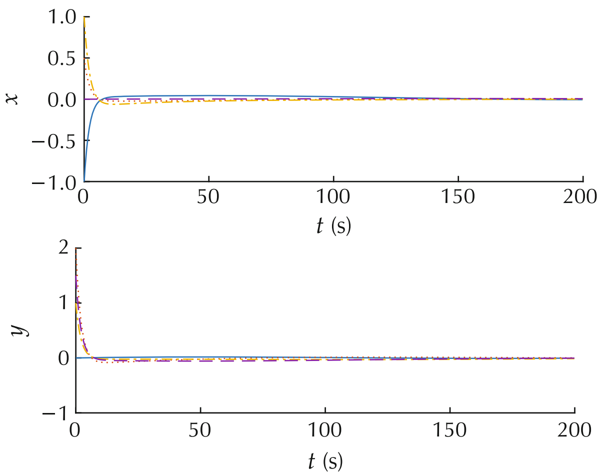
Fig. 1 $\gamma = 0.1$: $k_v = 0.5$, $k_g = k_{\mu p} = 0.0747$, $k_p = 0.00747$, $k_{\mu v} = 0.2986$. (a) Plane trajectories. (b) Maximum distance error. (c) Velocity evolution. (d) Acceleration evolution.



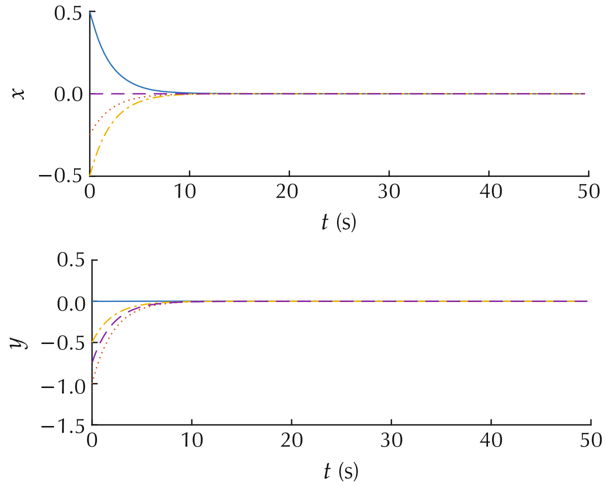
(a)



(b)



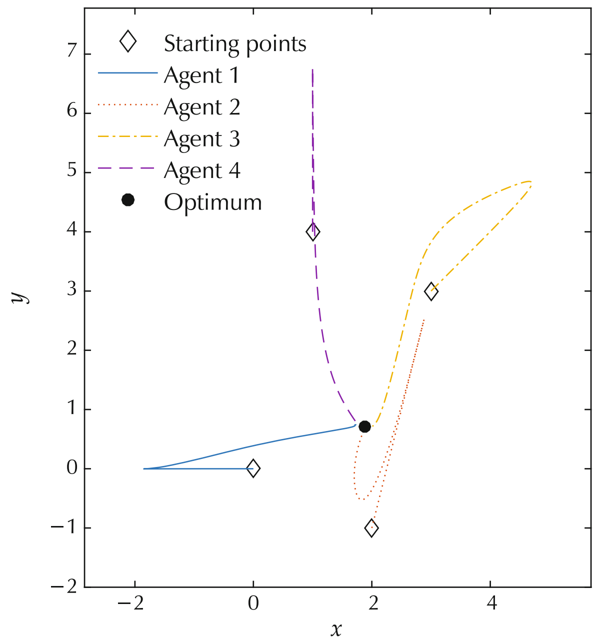
(c)



(d)

Fig. 2 $\gamma = 0.1$: $k_v = 0.5$, $k_g = k_{\mu p} = 0.0269$, $k_p = 0.00269$, $k_{\mu v} = 0.1076$. (a) Plane trajectories. (b) Maximum distance error. (c) Velocity evolution. (d) Acceleration evolution.

Intuitively, on the premise of exponential convergence, a larger γ assigns more weight on individual gradient descent and less on consensus terms, and hence we can expect a longer time to overcome the state difference among agents; on the other hand, it also implies that each agent's trajectory suffers from less interference of neighbors, especially the error integration, which results in a smoother trajectory.



(a)

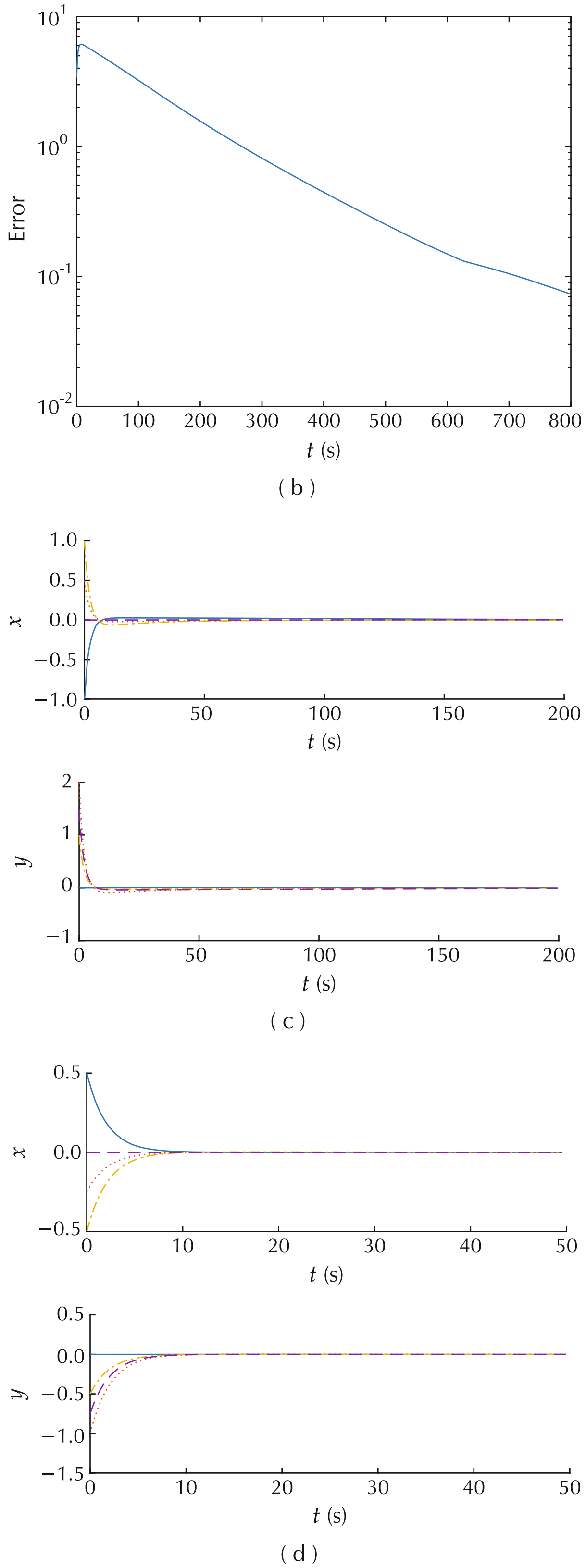


Fig. 3 $\gamma = 0.1$: $k_v = 0.5$, $k_g = k_{\mu p} = 0.0147$, $k_p = 0.00147$, $k_{\mu v} = 0.059$. (a) Plane trajectories. (b) Maximum distance error. (c) Velocity evolution. (d) Acceleration evolution.

6 Conclusions

In this paper the distributed optimal consensus problem of multiple double integrators under bounded velocity and acceleration was studied. Each agent is assigned a convex function as individual cost, and a bounded distributed control law was proposed to achieve consensus at the optimum of the aggregate cost under bounded velocity and acceleration. Specifically, the control input consists of a relative position error term, a damping term of its velocity, a gradient descent of its individual cost, as well as an integral feedback of the relative position and velocity error to correct the gradient differences. With the aid of quadratic Lyapunov functions, an exponentially fast convergence to the global optimum was established for strongly convex costs with Lipschitz continuous gradients, if the network is strongly connected and weight-balanced. Given the knowledge of the initial range of the optimum, the control gains were further properly chosen to satisfy the bounded constraints. A similar result holds when the relative velocity is not available. The case of EL dynamics was further discussed by the inverse dynamics control, assuming that the nonlinearity can be accurately known. In future we shall focus on the case when the a priori knowledge of the optimum is not available, or there exists uncertainty in the EL dynamics.

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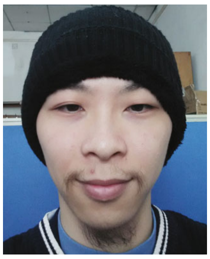
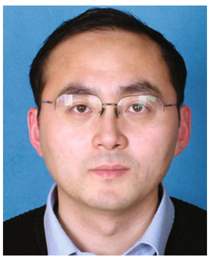


Zhirong Qiu received the B.Sc. and M.Sc. degrees in Applied Mathematics both from Sun Yat-Sen University, Guangzhou, China, and the Ph.D. degree from Nanyang Technological University (NTU), Singapore, in 2017. He is currently working as Research Fellow in the NTU-Delta Corporate Lab for Cyber-Physical Systems. His research interests include multi-agent systems and distributed optimization. E-mail: qiuz0005@e.ntu.edu.sg.

Lihua XIE received the B.E. and M.E. degrees in Electrical Engineering from Nanjing University of Science and Technology in 1983 and 1986, respectively, and the Ph.D. degree in Electrical Engineering from the University of Newcastle, Australia, in 1992. Since 1992, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a professor and Director, Delta-NTU Corporate Laboratory for Cyber-Physical Systems. He served as the Head of Division of Control and Instrumentation from July 2011 to June 2014. Dr Xie's research interests include robust control and estimation, networked control systems, multi-agent networks, and unmanned systems. He is currently an Editor-in-Chief of Unmanned Systems and has served as an editor of IET Book Series in Control and an Associate Editor of a number of journals including IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control Systems Technology, IEEE Transactions on Control of Network Systems, and IEEE Transactions on Circuits and Systems-II, etc. Dr Xie is a Fellow of IEEE, Fellow of IFAC, and a member of Board of Governors, IEEE Control System Society. E-mail: elhxie@ntu.edu.sg.



Yiguang HONG received his B.Sc. and M.Sc. both from Peking University, and his Ph.D. degree from Institute of Systems Science, Chinese Academy of Sciences (CAS). He is a Guan Zhaozhi Chair Professor of Academy of Mathematics and Systems Science, CAS. His current research interests include nonlinear control, multi-agent networks, distributed optimization and game, machine learning, and social networks. E-mail: yghong@iss.ac.cn.